Name: Solution Key   **PanthID:** __________

Midterm Exam - v.2   MAD 2104   Summer A 2015

To receive credit you MUST SHOW ALL YOUR WORK.

1. (10 pts) Write, in simple English, the negation of each of the following statements. Using logical connectors/quantifiers as an intermediate step may help. Do not start with a negation like "It is not true that ...".

(a) If you remember the password then you can login.

$\neg(p \to q) \equiv p \wedge \neg q$

You remember the password and you cannot login

(b) There is someone in this room who has traveled to all of the 50 states of the US.

For every person in this room there is a state in the US which that person has not visited

$\neg(\exists x \ \forall s \ V(x,s)) \equiv \forall x \ \exists s \ \neg V(x,s)$ $V(x,s)$: person x visited state s.

2. (10 pts) Let $A_i = \{i, i+1, i+2, ...\}$, for every positive integer i. Find the sets:

(a) $\bigcap_{i=1}^{+\infty} A_i = \emptyset$ since no positive integer is common to <u>all</u> sets A_i

$A_1 = \{1, 2, 3, \ldots\}$
$A_2 = \{2, 3, 4, \ldots\}$
$A_3 = \{3, 4, 5, \ldots\}$

(b) $\bigcup_{i=1}^{5} A_i = A_1 = \{1, 2, 3, \ldots\} = \mathbb{Z}^+$

Note that $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$

3. (10 pts) Show that $(p \to q) \to r$ is not logically equivalent with $p \to (q \to r)$.

see solution in version 1.

4. (10 pts) Use logic (including DeMorgan's laws for logic) to prove this DeMorgan's laws for sets: $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

$x \in \overline{A \cup B} \Leftrightarrow x \notin A \cup B \Leftrightarrow \neg(x \in A \cup B) \Leftrightarrow \neg(x \in A \vee x \in B) \Leftrightarrow$

$\Leftrightarrow \neg x \in A \wedge \neg x \in B \Leftrightarrow x \notin A \wedge x \notin B \Leftrightarrow x \in \overline{A} \wedge x \in \overline{B} \Leftrightarrow x \in \overline{A} \cap \overline{B}$

DeMorgan for logic

Thus $x \in \overline{A \cup B} \Leftrightarrow x \in \overline{A} \cap \overline{B}$, so $\overline{A \cup B} = \overline{A} \cap \overline{B}$

5. (10 pts) Consider the relation $\mathcal{R}$ on the set of all real numbers $\mathbf{R}$ defined by $(x, y) \in R$ if and only if $x - y$ is an integer.

(8 pts) Show that $\mathcal{R}$ is an equivalence relation on the set real numbers $\mathbf{R}$. Briefly justify each property.

(2 pts) What is the equivalence class of $\sqrt{5}$?

See solution in version I

$[\sqrt{5}] = \{\sqrt{5} + k \mid k \in \mathbb{Z}\}$.

6. (10 pts) Let $A = \{a, b, c\}$. If any part of this problem is impossible, explain why.

(a) Give an example of a relation $\mathcal{R}$ on A which is neither symmetric nor anti-symmetric.

One of the possible examples:

$R = \{(a,b), (b,a), (a,c)\}$

(b) Give an example of a relation $\mathcal{S}$ on A which is reflexive, symmetric, but not transitive.

One of the possible examples:

$\cancel{R}= \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c), (c,b)\}$

7. (16 pts) Determine if each of the following statements is True or False. No justification needed, just the answer will be graded.

T (a) A conditional statement is logically equivalent to its contrapositive.

T (b) For any set A, $A \in \mathcal{P}(A)$.

F (c) Any square $n \times n$ board can be tiled with dominoes (for all $n \geq 2$).

F (d) If the domain for a, b is the set of all real numbers, then $\forall a\, \exists b,\ ab = 2$. If $a = 0$, $\nexists b$ s.t. $0 \cdot b = 2$

T (e) $\exists a, b \in \mathbf{N},\ |a^2 - b^3| = 1$

F (f) $\exists x(P(x) \wedge Q(x)) \equiv (\exists x P(x)) \wedge (\exists x Q(x))$

T (g) $\exists x(P(x) \vee Q(x)) \equiv (\exists x P(x)) \vee (\exists x Q(x))$

T (h) The function $f(x) = 2x + 1$ is a bijection from $\mathbf{R}$ to $\mathbf{R}$.

8. (10 pts) We would like to determine the relative salaries of three coworkers, Fred, Maggie, Janice, using the following facts. First, we know that there are no ties in their salaries. Then we know that if Fred is not the highest paid of the three, then Janice is. Finally, we know that if Janice is not the lowest paid, then Maggie is paid the most. Is it possible to determine the relative salaries of Fred, Maggie and Janice from what we know? If so, who is paid the most and who the least ? Explain your reasoning in words (for max credit, this should be a logical sequence of deductions, with clear reasons).

see solution on version 1

9. Choose ONE (note the different values). If you choose (b) or (c), you may use without proof statement (a).

(a) (12 pts) Prove that $\sqrt{2}$ is not rational.

(b) (8 pts) Prove or disprove: There exist a rational number x and a rational number y so that x^y is irrational.

(c) (12 pts) Prove or disprove: There exist an irrational number x and an irrational number y so that x^y is rational.

See solutions in v.1

10. Choose ONE (note the different values):

(a) (8 pts) Prove that if $f : A \to B$ and $g : B \to C$ are both onto functions then the composition $g \circ f : A \to C$ is also onto.

(b) (12 pts) Prove that if $f : A \to B$ is an one-to-one function then for any subsets $S \subseteq A$, $T \subseteq A$,

$$f(S) \cap f(T) \subseteq f(S \cap T)\,.$$

see solutions in v.1