Name: Solution Key PanthID: ____________

Quiz 4 – take home MAD 2104 Summer A 2015

Due Monday, June 8. For full credit, you must show all your work.

1. (6 pts) This is a counting problem related to the one you did in class today. In each case, it is easier to first count the complement set. There are 26 English letters, with 5 vowels and 21 consonants. It's OK if you just give the answers for this problem, but it may be helpful to you to write in words what the complement set is.

(i) How many strings of eight uppercase English letter are there that contain at least one vowel, if letters can be repeated? The complement set is the set of strings of 8 letters which contain no vowels (so all consonants)

$26^8 - 21^8$

(ii) How many strings of eight uppercase English letter are there that contain at least one vowel, if letters cannot be repeated.

$P(26,8) - P(21,8)$ again complement rule, but now letters are not repeated.

2. (10 pts) Consider the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix}, \quad \text{where } d \text{ is a given constant.}$$

(a) Compute A^2, A^3, A^4, and then guess a formula for A^n.
(b) Use mathematical induction to prove your formula for A^n.

(a) $A^2 = \begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 1+d \\ 0 & d^2 \end{pmatrix}$ $A^3 = A^2 \cdot A = \begin{pmatrix} 1 & 1+d \\ 0 & d^2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 1+d+d^2 \\ 0 & d^3 \end{pmatrix}$

$A^4 = A^3 \cdot A = \dots = \begin{pmatrix} 1 & 1+d+d^2+d^3 \\ 0 & d^4 \end{pmatrix}$ Guess $A^n = \begin{pmatrix} 1 & 1+d+d^2+\dots+d^{n-1} \\ 0 & d^n \end{pmatrix}$ for all $n \geq$

(b) Basic Step $A^1 = \begin{pmatrix} 1 & 1 \\ 0 & d^1 \end{pmatrix}$ ✓

Inductive Step: Assume $A^n = \begin{pmatrix} 1 & 1+d+\dots+d^{n-1} \\ 0 & d^n \end{pmatrix}$. Then

$A^{n+1} = A^n \cdot A = \begin{pmatrix} 1 & 1+d+\dots+d^{n-1} \\ 0 & d^n \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 1 & 1+d(1+d+\dots+d^{n-1}) \\ 0 & d^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1+d+\dots+d \\ 0 & d^{n+1} \end{pmatrix}$

So Q.E.

3. (10 pts) Use strong induction to show that every positive integer n can be written as a sum of **distinct** powers of 2, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. For example, for $n = 23$, $23 = 2^4 + 2^2 + 2^1 + 2^0$.

Hint: One way to establish the inductive step is the following: Assuming the statement is true for all integers up to n, to get it for $n+1$ argue as follows: $n+1$ must fall between two successive powers of 2 (why?). That is, there exists k integer, such that $2^k \leq n+1 < 2^{k+1}$. Then consider $n+1-2^k$ and apply the inductive assumption.

Solution Pb. 3 Quiz 4: We will use Strong Induction.

Let $P(n)$ be the statement that n can be written as sum of distinct powers of 2, where n is a fixed positive integer.

Basic Step: $n=1$

$1 = 2^0$ so $P(1)$ is true.

Assume next that $P(j)$ is true for all $1 \le j \le n$.

We want to show that $P(n+1)$ is true.

Following the hint, $\exists$ ~~positive~~ positive integer ~~k~~ so that

$2^k \le n+1 < 2^{k+1}$ (because 2^k grows without bound, so there will be a smallest k so that $n+1 < 2^{k+1}$)

Consider now $n+1-2^k$. By the double inequality above

$$0 \le n+1-2^k < 2^{k+1} - 2^k = 2^k$$

If $n+1-2^k = 0 \Rightarrow n+1 = 2^k$ so ~~we are done~~ $P(n+1)$ is true.

If $0 < n+1-2^k < 2^k \le n+1 \Rightarrow 1 \le n+1-2^k \le n$, so by the strong induction assumption $n+1-2^k$ can be written as a distinct sum of powers of 2. Note that none of the terms in this sum can be 2^k as $n+1-2^k < 2^k$.

Thus $n+1 = 2^k + $ (~~sum~~ distinct of lower powers of 2).

So we proved that ~~$\forall n$~~ ($P(j)$ true for $1 \le j \le n$) $\to P(n+1)$

So by strong induction, $P(n)$ is true for all n.