Exam 1   MAA 3200   Fall 2009

1. (12 pts) Is the following propositional expression a tautology? A contradiction? Justify your answer.

$$[(P \to Q) \vee [Q \to R]] \to (P \to R)$$

2. (14 pts) Consider the following statement:

" $x^2 + y^2 + z^2$ cannot be of the form $8k + 7$, with $k \in \mathbb{Z}$, when x, y and z are odd."

(a) Rewrite the statement using quantifiers.

(b) Write an useful negation (using words) of the initial statement.

3. (12 pts) Prove that $\sqrt{2} \notin \mathbb{Q}$.

4. (12 pts) How many pairs of primes p and q are there such that $p - q = 7$? Prove your answer.

5. (12 pts) Prove by induction that that for each natural number $n \geq 1$

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left[\frac{n(n+1)}{2} \right]^2.$$

6. (16 pts) Let A, B, C denote arbitrary sets. Prove or find a counter-example to each of the following:

(a) $(A \setminus B) \bigcup C \subseteq A \setminus (B \bigcup C)$ (b) $A \setminus (B \bigcup C) \subseteq (A \setminus B) \bigcup C$

7. (20 pts) Consider the statement: "Any natural number greater or equal to 10 can be written as a sum of numbers, each of each is either a 5 or a 7".

(a) (5 pts) Rewrite the statement using quantifiers and symbols.

(b) (5 pts) What is the negation of the statement? (Answer either in words or symbols is ok.)

(c) (5 pts) Do you think the original statement is true, false, or neither? Briefly justify your answer.

(d) (5 pts) Is there any flaw in the following proof by induction of the original statement?
Proof: Let

$$S = \{n \in \mathbb{N} \mid n \geq 10, \ n \text{ can be written as a sum numbers, each of each a 5 or 7}\}$$

We'll proceed by strong induction on n.
Basic step: $n = 10 = 5 + 5$, so $10 \in S$.
Assume now that $10, ..., n \subset S$. We want to prove that $n + 1 \in S$. But $n + 1 = 5 + (n - 4)$. Since $n - 4 \leq n$, by the inductive assumption $n - 4 \in S$. Thus $n - 4$ can be written as a sum of 5's and 7's and because $n + 1 = 5 + (n - 4)$, $n + 1$ will have the same property. Thus $n + 1 \in S$. By (extended) strong induction,

$$S = \{n \in \mathbb{N} \mid n \geq 10\},$$

and the statement is proved.

8. (12 pts) Prove by induction that the sum of any three consecutive positive cubes is a multiple of 9. (For example, $3^3 + 4^3 + 5^3 = 216 = 9 \cdot 24$.)