Exam 2 - MAS 3105 Spring06 **NAME:** Answer Key

To receive credit you MUST SHOW ALL YOUR WORK.

1. (15 pts) Define each of the following (for (c), suppose S a subspace of $\mathbf{R}^n$):

(a) similar matrices (b) linear transformation (between two vector spaces) (c) $S^\perp$

See class notes or textbook.

2. (25 pts) Answer True or False. Briefly (but correctly) justify your answer (5 pts each).

(a) If A similar to B then A^2 is similar to B^2.

True. $A \sim B$ implies $B = S^{-1}AS$, for some non-singular matrix S. Then $B^2 = S^{-1}ASS^{-1}AS = S^{-1}AIAS = S^{-1}A^2S$, thus $A^2 \sim B^2$.

(b) For any matrix $A \in \mathcal{M}_{mn}$, $N(A) \oplus Range(A^T) = \mathbf{R}^n$.

True. From Thm 5.2.1. we know $N(A) = Range(A^T)^\perp$, but then we also know that $\mathbf{R}^n = S \oplus S^\perp$ for any subspace S. We can just take $S = Range(A^T)$.

(c) $\|\mathbf{u} + \mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, for any two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^2$.

False. It is very easy to find counter-examples, as the equality holds only when the vectors are collinear. Otherwise $\|\mathbf{u} + \mathbf{v}\| < \|\mathbf{u}\| + \|\mathbf{v}\|$ from the triangle inequality (sum of the lengths of two sides of a triangle is strictly larger then the third).

(d) For any matrix A, the rank of A is equal to the number of columns of A.

False. Rank of A is equal to the number of *independent* columns of A.

(e) If $\mathbf{x}_1 \perp \mathbf{x}_2$ and $\mathbf{x}_2 \perp \mathbf{x}_3$, then $\mathbf{x}_1 \perp \mathbf{x}_3$, for any $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbf{R}^3$.

False. For instance, let $\mathbf{x}_1 = \mathbf{e}_1$, $\mathbf{x}_2 = \mathbf{e}_2$, $\mathbf{x}_3 = \mathbf{e}_3 + \mathbf{e}_1$ (many other examples are possible).

3. (15 pts) The following two matrices are row equivalent

$$A = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 2 & 1 & 4 \\ 2 & 4 & 1 & 5 \end{pmatrix} \quad \text{and } U = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (3 pts) Find a basis for the row space of A.

The first two rows of A are a basis for $Row(A)$. Equally good answer is to say that the first two rows of U are a basis, since $Row(A) = Row(U)$.

(b) (3 pts) Find a basis for the column space of A.

First and third column of A are a basis for $Col(A)$ since the first and third column of U contain pivots. It would be wrong to say that the first and third column of U are a basis for $Col(A)$, since $Col(A)$ and $Col(U)$ may be completely different.

(c) (3 pts) Find a basis for the null space of A.

The system $A\mathbf{x} = \mathbf{0}$ is equivalent to $U\mathbf{x} = \mathbf{0}$. Solving this (free variables are x_2 and x_4), we get $x_1 = -2s - t, x_2 = s, x_3 = -3t, x_4 = t$. Hence, a basis for $N(A)$ is $\{(-2, 1, 0, 0)^T, (-1, 0, -3, 1)^T\}$.

(d) (3 pts) Find the rank of A.

Rank(A) = dim(Row(A)) = dim(Col(A)) = 2.

(e) (3 pts) Find the nullity of A.

null(A) = dim(N(A)) = 2 (or null(A) = n - rank(A) = 4 - 2 = 2).

4. (10 pts) Let L be the operator on P_3 defined by $L(p(x)) = xp'(x) + p(1)$. Find the matrix A representing L with respect to $[1, x, x^2]$.

$L(1) = 1 = 1 + 0 \cdot x + 0 \cdot x^2$
$L(x) = x + 1 = 1 + 1 \cdot x + 0 \cdot x^2$
$L(x^2) = 2x^2 + 1 = 1 + 0 \cdot x + 2 \cdot x^2$
Thus the matrix of L with respect to the basis $[1, x, x^2]$ is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

5. (15 pts) Let $\mathbf{u}_1 = (1, 2)^T$, $\mathbf{u}_2 = (2, 5)^T$.

(a) (5 pts) Find the angle between $\mathbf{u}_1$ and $\mathbf{u}_2$.

From $\mathbf{u}_1 \cdot \mathbf{u}_2 = \|\mathbf{u}_1\|\|\mathbf{u}_2\| \cos\theta$, we get $\theta = \arccos(12/\sqrt{5 \cdot 29})$.

(a) (5 pts) Find the transition matrix corresponding to the change of basis from $[\mathbf{e}_1, \mathbf{e}_2]$ to $[\mathbf{u}_1, \mathbf{u}_2]$.

It's U^{-1}, where U is the matrix with columns $\mathbf{u}_1$ and $\mathbf{u}_2$. Computing we get

$$U^{-1} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

(b) (5 pts) Find the coordinates of $\mathbf{v} = (-1, 1)^T$ with respect to $[\mathbf{u}_1, \mathbf{u}_2]$.

$U^{-1}\mathbf{v} = (-7, 3)^T$.

6. (15 pts) Choose ONE of these to prove:

(a) If S is a subspace of $\mathbf{R}^m$, then $(S^\perp)^\perp = S$.

See textbook or class notes.

(b) Let S be a subspace of $\mathbf{R}^m$. For each $\mathbf{b} \in \mathbf{R}^m$, there is a unique element $\mathbf{p} \in S$ that is closest to $\mathbf{b}$, that is,

$$\|\mathbf{b} - \mathbf{y}\| > \|\mathbf{b} - \mathbf{p}\| \quad \text{for any } \mathbf{y} \neq \mathbf{p} \text{ in } S.$$

Furthermore, the vector $\mathbf{p}$ in S that is closest to $\mathbf{b} \in \mathbf{R}^m$ has the property $\mathbf{b} - \mathbf{p} \in S^\perp$.

See textbook or class notes.

7. (15 pts) Choose ONE of these to prove:

(a) If A is an $m \times n$ matrix, then

$$(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (A^T\mathbf{v}), \quad \text{for any vectors } \mathbf{u} \in \mathbf{R}^n \text{ and } \mathbf{v} \in \mathbf{R}^m.$$

$$Proof: \quad (A\mathbf{u}) \cdot \mathbf{v} = (A\mathbf{u})^T\mathbf{v} = \mathbf{u}^T A^T \mathbf{v} = \mathbf{u} \cdot (A^T\mathbf{v})$$

You may also say a few words to justify that all the matrix multiplications make sense (dimensions match).

(b) If A is an $m \times n$ matrix then $N(A^TA) = N(A)$.

See class notes. This is part of Ex. 13, section 5.2, which we solved in class.