Problems for Fermat's and Euler's Theorems

1. If p is prime, then, for all a, $a^p \equiv a \pmod{p}$.

2. What can you say about n and m if:

(a) $n^{96} \equiv m \pmod{17}$?

(b) $n^9 \equiv m \pmod{19}$?

3. (a) Show: if $7 \nmid n$, then $7 \mid (n^{12} - 1)$.

(b) Show: $n^{13} - n$ is divisible by 2, 3, 5, 7, 13, for all natural numbers n.

4. (a) Find the remainder of the sum $1 + a + a^2 + a^3 + ... + a^9$ mod 11, for each number $a < 11$. Can you explain the outcome?

(b) Make a theorem generalizing the statement from part (a) and prove your theorem.

5. Let $N = \overline{111...11}$, where N is a number in base 10, made up of p 1's and where p is a prime other than 3. Show that $N \equiv 1 \pmod{p}$.

6*. Let a, b be natural numbers such that $gcd(a, b) = 1$. Show that there exist numbers $m, n \in \mathbf{N}^*$ such that

$$a^m + b^n \equiv 1 \pmod{ab}.$$