Name: Solution key Panther ID: ____________

Exam 1 **MAP 2302: Summer B 2019**

1. (10 pts) Label each 1st order differential equation below with its type - exact, separable, homogeneous (as in section 2.2B), linear, Bernoulli (and not linear), or 'none of the above'. You **DO NOT** have to solve any of them.

(a) $\frac{dy}{dx} + \frac{2y}{x} = xy^3$ Bernoulli

(b) $(y^2+1)\,dx + (x^2+4)\,dy = 0$ separable

(c) $(x^2+xy+y^2)\,dx - (x^2+y^2)\,dy = 0$ homogeneous

(d) $\frac{dy}{dx} + \frac{y}{x^2} = \cos(y)$ none of the above

(e) $\frac{dy}{dx} + \frac{y}{x^2} = \cos x$ linear

2. (15 pts) Determine the constant A so that $\overbrace{(Ax^2y + 2y^2)}^{M}\,dx + \overbrace{(x^3 + 4xy)}^{N}\,dy = 0$ is exact. Solve the resulting DE (implicit solution is OK).

$$\frac{\partial M}{\partial y} = Ax^2 + 4y \qquad \frac{\partial N}{\partial x} = 3x^2 + 4y$$

so the D.E is exact $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Leftrightarrow A=3$

For $A=3$, we want to find a potential function $F(x,y)$ so that

$$\frac{\partial F}{\partial x} = 3x^2y + 2y^2 \quad \text{and} \quad \frac{\partial F}{\partial y} = x^3 + 4xy$$

$$\Downarrow$$

$$F(x,y) = \int (3x^2y + 2y^2)\,\partial x = x^3y + 2xy^2 + g(y)$$

From $\frac{\partial F}{\partial y} = x^3 + 4xy$ we get $x^3 + 4xy + g'(y) = x^3 + 4xy$

so $g'(y) = 0$ so $g(y) = c$

Thus, the implicit-solution of the D.E. when $A=3$ is

$$\boxed{x^3y + 2xy^2 = c}$$

3. (15 pts) Consider the differential equation $(\frac{dy}{dx})^2 - 4y = 0$.

(a) (6 pts) Check that the DE above has a one-parameter family of solutions of the form $y = (x+c)^2$.

From $y = (x+c)^2$, get $\frac{dy}{dx} = 2(x+c)$ so

$\left(\frac{dy}{dx}\right)^2 = 4(x+c)^2 = 4y$. Thus $y = (x+c)^2$ is a family of solutions

(b) (5 pts) Does the DE above have a unique solution such that $y(1) = 1$? Briefly justify.

No! Even in the family of solutions found in (a) imposing the initial condition get $1 = (1+c)^2$ so $1+c = \pm\sqrt{1} = \pm 1$ so $\underline{c=0}$ or $\underline{c=-2}$

Thus, both $y(x) = x^2$ and $y(x) = (x-2)^2$ are solutions satisfying $y(1) = 1$.

(c) (4 pts) Is part (b) in contradiction with the fundamental theorem for existence and uniqueness of 1st order IVP? Briefly justify.

No, there is no contradiction because the above D.E. is $\underline{\underline{NOT}}$ in the standard form $\frac{dy}{dx} = F(x,y)$, so the fundamental theorem does not apply!

4. (15 pts) Solve the initial value problem: $(x+y)\,dx - x\,dy = 0$, with $y(1) = 3$.

The ~~equation~~ D.E. is both homogeneous and linear, so you can choose the way to proceed. I'll write here the homogeneous technique, which is what most of you picked.

$$\frac{dy}{dx} = \frac{x+y}{x} \quad \text{or} \quad \frac{dy}{dx} = 1 + \frac{y}{x} \quad (*)$$

Let $\frac{y}{x} = v$ or $y = x \cdot v$. Then $\frac{dy}{dx} = 1 \cdot v + x\frac{dv}{dx}$ so

$(*)$ becomes $\cancel{v} + x\frac{dv}{dx} = 1 + \cancel{v} \iff dv = \frac{1}{x}dx$ separable D.E.

$$\int dv = \int \frac{1}{x}dx \Rightarrow v(x) = \ln|x| + c$$

Thus $y(x) = x\ln|x| + c \cdot x$

Imposing the initial condition $y(1) = 3$ get

$$3 = 1 \cdot \ln 1 + c \Rightarrow c = 3$$

Thus, the solution of the I.V.P is $\boxed{y(x) = x\ln x + 3x}$ for $\underline{x>0}$.

5. (16 pts) These are True/False questions. Answer and give a brief justification. Interpret *a good way* to mean *a standard efficient way.* (4 pts each).

(a) The equation $y'' + x^2y' + 4xy = e^x$ is a linear, 2nd order, ODE. **True** **False**

Justification: The above D.E. is of the form

$a_2(x)\, y'' + a_1(x)\, y' + a_0(x)\, y = b(x)$ which is the general form of a linear, 2nd order D.E.

(b) A good way to approach the DE $(x + 2y + 3)\, dx + (3x + 6y - 2)\, dy = 0$ is to set $z = x + 2y$, to make it separable.
True **False**

Justification: Case 2 of Thm. 2.7 (special transformations - section 2.4 B)
Can also see directly that the sub $z = x + 2y$ will transform the D.E. into a separable one.

(c) A good way to approach the DE $\frac{dy}{dx} + x^3y = xy^3$ is to set $y = vx$ to make it separable. **True** **False**

Justification: The D.E. is not homogeneous. It is Bernoulli with $n=3$, so one should set $v = y^{1-3} = y^{-2}$ to get a linear D.E.

(d) The IVP $\frac{dy}{dx} = \frac{x}{y}$, $y(1) = 1$, has a unique solution defined on an interval $(1 - \delta, 1 + \delta)$. **True** **False**

Justification: As $F(x,y) = \frac{x}{y}$ and $\frac{\partial F}{\partial y} = -\frac{x}{y^2}$ are both continuous in an open ball around $(1,1)$, the fundam. Thm. for existence & uniqueness for 1st order IVP applies.

6. (10 pts) A stone weighing 4 lbs falls from rest towards ground from a great height. As it falls, the air resistance is numerically equal to $v/2$ (in lbs), where v is the velocity (in feet per second). Set up an IVP in v and t that would allow you to solve for $v(t)$ (but you DO NOT have to solve it). You can use standard formulas such as $F = ma$ and $g = 32$ ft/sec^2.

Choosing an axis with positive orientation downward

$ma = mg - \frac{v}{2}$

$m\frac{dv}{dt} = mg - \frac{v}{2} \Leftrightarrow \frac{dv}{dt} = g - \frac{v}{2m}$

Since the weight $= mg = 4$ lbs. $\Rightarrow m = \frac{4}{g} = \frac{4}{32} = \frac{1}{8}$ slugs.

Thus $\frac{dv}{dt} = 32 - \frac{v}{2 \cdot \frac{1}{8}} \Leftrightarrow \boxed{\begin{cases} \frac{dv}{dt} = 32 - 4v \\ v(0) = 0 \end{cases}}$ ← The I.V.P. for the above problem.

7. (15 pts) Solve the following linear 1st order DE

$$\frac{dy}{dx} + \frac{2y}{x} = 4x$$

We should multiply by the integrating factor

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$$

$$x^2 \frac{dy}{dx} + x^2 \frac{2y}{\cancel{x}} = 4x^3$$

$$\frac{d}{dx}\left(x^2 \cdot y\right) = 4x^3$$

$$x^2 \cdot y = \int 4x^3\,dx$$

$$x^2 \cdot y = x^4 + c$$

so $\boxed{y(x) = x^2 + c \cdot x^{-2} = x^2 + \frac{c}{x^2}}$

is the family of solutions for the 1st order linear D.E.

8. (12 pts) Choose ONE to prove. If you have time and do both proofs, the second proof may give some (small) bonus towards a problem where you scored lower.

(A) State and prove the theorem stating that a certain substitution will change a general (non-linear) Bernoulli equation into a linear 1st order DE.

(B) Find (with proof) a condition on M, N that insures that a differential equation $M(x,y)dx + N(x,y)dy = 0$ has an integrating factor of the type $\mu = \mu(x)$.

see notes or textbook