Name: Solution Key Panther ID: ____________

Exam 2 **MAP 2302: Summer B 2018**

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Please turn your cell phone off at the beginning of the exam and place it in your bag, NOT in your pocket.
3. No electronic devices (cell phones, headphones, calculators of any kind, etc.) should be used at any time during the examination. Notes, texts or formula sheets should NOT be used either. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor. Violations will lead to a score of 0 on this exam.

1. (25 pts) These are True/False questions. Circle your answer and give a brief justification (5 pts each).

(a) If y_1 and y_2 are particular solutions of a 3rd order linear homogeneous DE with variable coefficients, then $y_1 - y_2$ is also a solution.

True **False**

Justification: For a homogeneous linear DE any linear combination of solutions is also a solution

(b) The Variation of Parameters method can be applied to find a particular solution of $y'' - 2y' + y = xe^x \ln x$.

True **False**

Justification: The complementary homogeneous equation has general solution $y_c = c_1 e^x + c_2 x \cdot e^x$ so we could look for a particular solution of the form $y_p = c_1(x) e^x + c_2(x) x \cdot e^x$ The UC method would not apply as $xe^x \ln x$ is not a UC function

(c) If f_1, f_2, f_3 are all solutions for $(x^2+1)y'' + (x-1)y' + (x+3)y = 0$, then $\{f_1, f_2, f_3\}$ are linearly dependent.

True **False**

Justification: If $\{f_1, f_2\}$ are linearly indep. then they form a fundam. set of solutions, so f_3 must be a linear combination of $\{f_1, f_2\}$

(d) The Wronskian of $\{e^{2x}, e^{-2x}\}$ is always non-zero. **True** **False**

Justification:

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \neq 0 \text{ for all } x.$$

(e) If a free undamped motion satisfies $x(t) = 3\sin(2t) + 4\cos(2t)$ then it oscillates with an amplitude exceeding 4.823.

True **False**

Justification:

The amplitude is $c = \sqrt{3^2+4^2} = 5 > 4.823$

2. (20 pts) (a) (6 pts) Find the general solution $y(t)$ of $y'' - 5y' + 4y = 0$.

Charact. equ. $\lambda^2 - 5\lambda + 4 = 0 \Leftrightarrow (\lambda - 4)(\lambda - 1) = 0 \Leftrightarrow \lambda_1 = 1, \lambda_2 = 4$

$$\boxed{y_c(t) = c_1 e^{t} + c_2 e^{4t}}$$

(b) (8 pts) Find the general solution of $y'' - 5y' + 4y = 3e^{2t}$.

Apply UC method and look for a particular solution $y_p(t) = Ae^{2t}$

Then $y_p'(t) = 2Ae^{2t}$, $y_p''(t) = 4Ae^{2t}$ so, plugging in,

$4Ae^{2t} - 10Ae^{2t} + 4Ae^{2t} = 3e^{2t} \Rightarrow -2A = 3 \Rightarrow A = -\frac{3}{2}$

Thus $y_p(t) = -\frac{3}{2}e^{2t}$ so the general solution of the equation is $\boxed{y(t) = c_1 e^{t} + c_2 e^{4t} - \frac{3}{2}e^{2t}}$

(c) (6 pts) Use the UC method to write the form for a particular solution of $y'' - 5y' + 4y = 5\sin(t) + 3te^{4t} + 10$ including constants A, B, C etc as needed. You do **NOT** have to compute the constants for this part.

$$y_p(t) = A\sin t + B\cos t + \underbrace{(Ct^2 + Dt)}e^{4t} + E$$

adjusted because of the "interference" with $y_c(t)$

homogeneous

3. (15 pts) Find the solution of the IVP: $9x^2y'' + 3xy' + y = 0, \quad y(1) = 3, \; y'(1) = 2$.

It's a Cauchy-Euler DE, so you could either do the sub $x = e^t$, or look for solutions of the form $y = x^r$.

I write below the solution via the second method.

$$y = x^r \Rightarrow y' = r x^{r-1} \Rightarrow y'' = r(r-1) x^{r-2}$$

The equation becomes

$$9r(r-1)x^r + 3r x^r + x^r = 0 \quad \text{or}$$

$$9r^2 - 9r + 3r + 1 = 0 \Leftrightarrow 9r^2 - 6r + 1 = 0 \Leftrightarrow (3r-1)^2 = 0$$

so we get a double root $r_1 = r_2 = \frac{1}{3}$.

In this case a fundamental set of solutions is

$$y_1(x) = x^{\frac{1}{3}} \quad \text{and} \quad y_2(x) = x^{\frac{1}{3}} \ln x$$

The general solution is $y(x) = c_1 x^{\frac{1}{3}} + c_2 x^{\frac{1}{3}} \ln x$

and the constants c_1, c_2 are determined from the initial conditions.

$$y'(x) = \frac{1}{3} c_1 x^{-\frac{2}{3}} + c_2\left(\frac{1}{3} x^{-\frac{2}{3}} \ln x + x^{\frac{1}{3}} \cdot \frac{1}{x}\right)$$

$$y'(x) = \frac{1}{3} c_1 x^{-\frac{2}{3}} + c_2\left(\frac{1}{3} x^{-\frac{2}{3}} \ln x + x^{-\frac{2}{3}}\right)$$

$$\text{So} \begin{cases} y(1) = 3 \\ y'(1) = 2 \end{cases} \Leftrightarrow \begin{cases} c_1 = 3 \\ \frac{1}{3} c_1 + c_2 = 2 \end{cases} \Leftrightarrow \begin{cases} c_1 = 3 \\ c_2 = 1 \end{cases}$$

Thus the solution of the I.V.P. is

$$y(x) = 3x^{\frac{1}{3}} + x^{\frac{1}{3}} \ln x \qquad \text{for } x > 0.$$

4. (15 pts) An 8-lb weight stretches a hanging spring $\overbrace{\text{6 inches}}^{0.5\,ft}$ from its natural position. The weight is then pulled down another 6 inches and released at time $t = 0$. The medium offers resistance equal to $4x'$ where x' is the velocity in feet per sec. Find a formula for the displacement $x(t)$. Note that gravitational acceleration is $g = 32$ ft/sec^2.

$$8 = k \cdot 0.5 \Rightarrow k = \frac{8}{0.5} = 16$$

$$8 = m \cdot g \Rightarrow m = \frac{8}{32} = \frac{1}{4} \text{ slugs.}$$

$$\begin{cases} \frac{1}{4} x'' + 4x' + 16x = 0 \\ x(0) = 0.5 \quad x'(0) = 0 \end{cases}$$

D.E. is equiv. to $x'' + 16x' + 64x = 0$

$$\Leftrightarrow (\lambda + 8)^2 = 0 \qquad \lambda_1 = \lambda_2 = -8$$

General solution is

$$x(t) = c_1 e^{-8t} + c_2 t \cdot e^{-8t}$$

and we determine c_1, c_2 from the initial conditions.

$$x(0) = 0.5 \Rightarrow c_1 + 0 = 0.5$$

$$x'(t) = -8c_1 e^{-8t} + c_2 (1 \cdot e^{-8t} + t e^{-8t} (-8))$$

$$x'(0) = 0 \Rightarrow -8c_1 + c_2 = 0 \qquad \text{so } c_2 = 8c_1 = 8 \times 0.5 = 4$$

Thus $\boxed{x(t) = 0.5 e^{-8t} + 4t e^{-8t}}$

5. (15 pts) Given that $y = e^{2x}$ is a solution of $(2x+1)y'' - 4(x+1)y' + 4y = 0$, find a linearly independent solution by reducing the order. Write the general solution.

Hint: If you forgot the procedure, take a look on Problem 6 (A) (on next page).

Let $y = e^{2x} \cdot v$. Then $y' = 2e^{2x} \cdot v + e^{2x} \cdot v'$ and

$$y'' = 4e^{2x} \cdot v + 2e^{2x} \cdot v' + 2e^{2x} \cdot v' + e^{2x} \cdot v''$$

$$y'' = 4e^{2x} \cdot v + 4e^{2x} \cdot v' + e^{2x} \cdot v''$$, so substituting we get

$$(2x+1)(4e^{2x} \cdot v + 4e^{2x} \cdot v' + e^{2x} \cdot v'') - 4(x+1)(2e^{2x} \cdot v + e^{2x} \cdot v') + 4e^{2x} \cdot v = 0$$

$$\Leftrightarrow e^{2x}\Big[\big(\cancel{(8x+4)v} - \cancel{(8x+8)v} + \cancel{4v}\big) + \big((8x+4)v' - (4x+4)v'\big) + (2x+1)v''\Big] = 0 \quad \div e^{2x}$$

$$\Leftrightarrow 4x\,v' + (2x+1)v'' = 0$$

or, with $w = v'$, we get $(2x+1)w' = -4xw$.

This is separable $\dfrac{dw}{w} = -\dfrac{4x}{2x+1}dx$

$$\int \frac{dw}{w} = -\int \frac{4x+2-2}{2x+1}dx$$

$$\ln w = -\int\left(2 - \frac{2}{(2x+1)}\right)dx$$

$$\ln w = -2x + \ln(2x+1) \Rightarrow w = e^{-2x + \ln(2x+1)} = e^{-2x} \cdot (2x+1)$$

Thus $v' = (2x+1)e^{-2x}$ so $v(x) = \int (2x+1)e^{-2x}\,dx$.

Use I.B.P. with $du = e^{-2x}dx$ $\quad v = 2x+1$

$u = -\frac{1}{2}e^{-2x}$ $\quad dv = 2\,dx$

$$v(x) = -\frac{1}{2}(2x+1)e^{-2x} + \int e^{-2x}dx = -\frac{1}{2}(2x+1)e^{-2x} - \frac{1}{2}e^{-2x} = -(x+1)e^{-2x}$$

Thus, the second solution is $y_2(x) = y_1(x) \cdot v = e^{2x} \cdot (-(x+1) \cdot e^{-2x}) = -(x+1)$

The general solution is

$$\boxed{y(x) = c_1(x+1) + c_2 e^{2x}}$$

6. (15 pts) Choose ONE proof. If you have time to do both proofs, the second score may give some bonus towards a previous problem where your score is lower.

(A) Suppose $f(x)$ is a non-trivial solution of the second order homogeneous linear ODE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \, .$$

Show that the substitution $y = f(x)v$, followed by $w = v'$, will reduce the above ODE to a first order homogeneous linear ODE in w.

(B) Derive the formulas for $c_1'(x)$ and $c_2'(x)$ from the VP method.
That is, show that if y_1, y_2 are linearly independent solutions of $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then a particular solution for $a_2(x)y'' + a_1(x)y' + a_0(x)y = b(x)$ is given by

$$y_p(x) = c_1(x)y_1(x) + c_2(x)y_2(x) \, , \text{ where}$$

$$c_1'(x) = -\frac{b(x)y_2(x)}{a_2(x)w(x)} \, , \; c_2'(x) = \frac{b(x)y_1(x)}{a_2(x)w(x)} \text{ and } w(x) \text{ denotes the Wronskian of } y_1, y_2.$$

See textbook or class notes