The number e and compound interest

1. **Activity for you and your students.**

Suppose you inherit \$ 1 (If it is more exciting, you could also assume that " \$ 1" is really \$ 1,000,000, but work with it as 1). Suppose you also find a bank that pays an annual interest rate $r = 100\%$ and you deposit your inheritance there. How much money would you have in your account at the end of the year if:

(a) the interest is compounded yearly?

(b) the interest is compounded monthly?

(c) the interest is compounded daily?

(d) the interest is compounded n times a year?

(e) the interest is compounded continuously? (Hint: use a limit for this.)

2. A proof that you should see once.

Your answer for 1(e) should have been $\lim_{n\to+\infty}\left(1+\dfrac{1}{n}\right)^n$

This limit exists and, by definition, is the famous number $e = 2.718281828459045....$ But how does one show rigorously that the sequence $a_n = \left(1+\frac{1}{n}\right)^n$ converges? The calculus "proofs" with l'Hopital's rule are not really rigorous, as this limit is essentially used in proving that the derivative of $\ln x$ is $1/x$.

One shows that the sequence $\{a_n\}$ is bounded and increasing, hence convergent.

For $\{a_n\}$ bounded, we claim that $0 \le a_n \le 3$ for any $n \ge 1$.

The left inequality is obvious. To get the right, we use the binomial formula

$$a_n = 1 + C_n^1\frac{1}{n} + C_n^2\frac{1}{n^2} + C_n^3\frac{1}{n^3} + ... + C_n^n\frac{1}{n^n}$$

and note

$$C_n^1\frac{1}{n} = \frac{n}{1}\cdot\frac{1}{n} = 1\ ,\ C_n^2\frac{1}{n^2} = \frac{n(n-1)}{1\cdot 2}\cdot\frac{1}{n^2} \le \frac{1}{2}\ ,\ C_n^3\frac{1}{n^3} = \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot\frac{1}{n^3} \le \frac{1}{2^2}\ ,\ ...$$

Thus

$$a_n \le 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} = 3 - \frac{1}{2^{n-1}} \le 3$$

To show that $\{a_n\}$ is increasing, we'll use at a crucial point the Bernoulli inequality:

$(1+x)^n \ge 1+nx$, for any real number $x \ge -1$ and any natural number n.

A proof of this can be given by induction (Exercise!).

Consider the quotient

$$\frac{a_{n+1}}{a_n} = \frac{(n+2)^{n+1}\cdot n^n}{(n+1)^{2n+1}} = \frac{n+1}{n}\cdot\frac{(n+2)^{n+1}\cdot n^{n+1}}{(n+1)^{2n+2}} = \frac{n+1}{n}\cdot\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n+1},$$

which we finally rewrite as

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n}\cdot\left(1 - \frac{1}{(n+1)^2}\right)^{n+1}.$$

Applying now the Bernoulli inequality, we get

$$\frac{a_{n+1}}{a_n} \ge \frac{n+1}{n}\cdot(1 - \frac{n+1}{(n+1)^2}) = 1\ .$$

Thus, we proved that $a_{n+1} \ge a_n$ for all n, so the sequence $\{a_n\}$ is increasing.

3. One more exercise for you. Using now the limit

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e\ ,$$

show that if you deposit a principal P dollars in a bank account that pays an annual interest rate r compounded continuously, your balance in the account $B(t)$ after t years since the deposit is given by

$$B(t) = Pe^{rt}$$