Name: Solution Key **Panther ID:** ______

Exam 2 Trigonometry Summer A 2016

Important Rules:

1. Unless otherwise mentioned, to receive full credit you MUST SHOW ALL YOUR WORK. Answers which are not supported by work might receive no credit.
2. Turn off your cell phone at the beginning of the exam and place it in your bag, NOT in your pocket.
3. **No calculators, of any kind,** *are allowed. Any other electronic devices, notes, texts or formula sheets are also prohibited. Concentrate on your own exam. Do not look at your neighbor's paper or try to communicate with your neighbor.*

1. (8 pts) (a) (4 pts) State the domain and range for $y = \sin^{-1} x$ (or $y = \arcsin x$).

Domain $[-1,1]$; Range $[-\frac{\pi}{2}, \frac{\pi}{2}]$

(b) (4 pts) State the domain and range for $y = \tan^{-1} x$ (or $y = \arctan x$).

Domain $(-\infty, +\infty)$ Range $(-\frac{\pi}{2}, \frac{\pi}{2})$

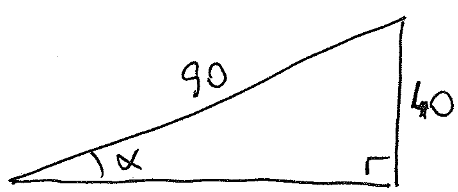
2. (12 pts) Find the exact value of each of the following expressions (4 pts each):

(a) $\sin^{-1} 1 = \frac{\pi}{2}$

(b) $\cos(85^o)\cos(25^o) + \sin(85^o)\sin(25^o) = \cos(85^\circ - 25^\circ) = \cos(60^\circ) = \frac{1}{2}$

(c) $\tan^{-1}\left(\tan\left(\frac{9\pi}{10}\right)\right) = \tan^{-1}\left(\tan\left(-\frac{\pi}{10}\right)\right) = -\frac{\pi}{10}$

3. (8 pts) A kite flies at a height of 40 feet above ground when 90 feet of string is out. If the string is in a straight line, find the angle that it makes to the ground. You may leave your answer as an inverse trigonometric function.



$\sin\alpha = \frac{40}{90} = \frac{4}{9}$

$\alpha = \arcsin\left(\frac{4}{9}\right)$

4. (12 pts) All parts of this problem are true or false questions. Circle the correct answer (2 pts each). No justification is required. No partial credit.

(a) For all x, $\cos^{-1} x = \sec x$. **True** **False**

(b) For all $x \in [-1, 1]$, $\arcsin x + \arccos x = \frac{\pi}{2}$. **True** **False**

(c) For all $x \in [-1, 1]$, $\sin(\sin^{-1} x) = x$. **True** **False**

(d) $\sin\left(x + \frac{\pi}{2}\right) = \sin x + \sin \frac{\pi}{2}$ is an identity. **True** **False**

(e) $\tan(\pi - x) = -\tan x$ is an identity. **True** **False**

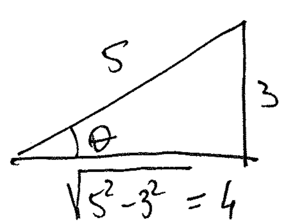
(f) The equation $2 \sin x + 3 = 0$ has no solution. **True** **False**

5. (12 pts) If $\sin\theta = \dfrac{3}{5}$, and θ is in quadrant II, use double angle formulas to find each of the following:

(a) $\sin(2\theta)$ $= 2\sin\theta\cos\theta = -\dfrac{24}{25}$

(b) $\cos(2\theta)$ $= \cos^2\theta - \sin^2\theta = \dfrac{7}{25}$

(c) $\tan(2\theta)$ $= \dfrac{\sin(2\theta)}{\cos(2\theta)} = -\dfrac{24}{7}$



$\cos\theta = -\dfrac{4}{5}$ since θ is in second quadrant

Thus

$$\sin(2\theta) = 2 \cdot \frac{3}{5} \cdot \left(-\frac{4}{5}\right) = -\frac{24}{25}$$

$$\cos(2\theta) = \left(-\frac{4}{5}\right)^2 - \left(\frac{3}{5}\right)^2 = \frac{7}{25}$$

$$\tan(2\theta) = \frac{-\frac{24}{25}}{\frac{7}{25}} = -\frac{24}{7}$$

6. (18 pts) Verify each identity (your work should be clear) (6 points each).

(a) $\sec^2 x \csc^2 x = \sec^2 x + \csc^2 x$

$$\text{Left-side} = \sec^2 x \csc^2 x = \frac{1}{\cos^2 x} \cdot \frac{1}{\sin^2 x} = \frac{1}{\sin^2 x \cos^2 x}$$

$$\text{Right side} = \sec^2 x + \csc^2 x = \frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x \cdot \sin^2 x} =$$

$$= \frac{1}{(\sin^2 x)\cdot(\cos^2 x)} = \text{Left-side}$$

(b) $(\tan^2 \theta + 1)(\cos^2 \theta + 1) = \tan^2 \theta + 2$

$$\cancel{\text{Left-side} = \left(\frac{\sin^2\theta}{\cos^2\theta} + 1\right)\left(\cos^2\theta + 1\right)}$$

Using $\sec^2 x = 1 + \tan^2 x$, $\text{left-side} = \sec^2\theta \cdot (\cos^2\theta + 1) =$

$$= \frac{1}{\cos^2\theta}\left(\cos^2\theta + 1\right) = 1 + \frac{1}{\cos^2\theta} = 1 + \sec^2\theta$$

$$\text{Right-side} = \tan^2\theta + 2 = \tan^2\theta + 1 + 1 = 1 + \sec^2\theta$$

Thus Left-side = Right-side

(c) $\tan\left(\frac{\pi}{4} - \theta\right) = \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}$

$$\text{Left-side} = \frac{\sin\left(\frac{\pi}{4} - \theta\right)}{\cos\left(\frac{\pi}{4} - \theta\right)} = \frac{\sin\frac{\pi}{4}\cos\theta - \cos\frac{\pi}{4}\sin\theta}{\cos\frac{\pi}{4}\cos\theta + \sin\frac{\pi}{4}\sin\theta} = \frac{\frac{\sqrt{2}}{2}\cos\theta - \frac{\sqrt{2}}{2}\sin\theta}{\frac{\sqrt{2}}{2}\cos\theta + \frac{\sqrt{2}}{2}\sin\theta} =$$

$$= \frac{\cancel{\frac{\sqrt{2}}{2}}\left(\cos\theta - \sin\theta\right)}{\cancel{\frac{\sqrt{2}}{2}}\left(\cos\theta + \sin\theta\right)} = \text{Right-side}$$

7. (20 pts) Solve each of the following equations on the interval $[0, 2\pi)$.

(a) (6 pts) $4\sin x + 1 = 2\sin x$

$2\sin x = -1$

$\sin x = -\frac{1}{2}$

so $x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$ or $x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$

(b) (6 pts) $\tan(2x) = -1$

All solutions are: $2x = \frac{3\pi}{4} + k\pi$ with $k \in \mathbb{Z}$.

or $x = \frac{3\pi}{8} + k\frac{\pi}{2}$ with $k \in \mathbb{Z}$

The solutions in $[0, 2\pi)$ are obtained when

$k=0$, $\boxed{x = \frac{3\pi}{8}}$ or $k=1$ $\boxed{x = \frac{3\pi}{8} + \frac{\pi}{2} = \frac{7\pi}{8}}$

or $k=2$, $\boxed{x = \frac{3\pi}{8} + \cancel{2}\cdot\frac{\pi}{\cancel{2}} = \frac{11\pi}{8}}$ or $k=3$ $\boxed{x = \frac{3\pi}{8} + \frac{3\pi}{2} = \frac{15\pi}{8}}$

(c) (8 pts) $4\sin^2 x + 4\cos x - 5 = 0$

We should use $\sin^2 x + \cos^2 x = 1$ to replace $\sin^2 x$ with $1 - \cos^2 x$.

$4(1-\cos^2 x) + 4\cos x - 5 = 0$ so $4 - 4\cos^2 x + 4\cos x - 5 = 0$

so $-4\cos^2 x + 4\cos x - 1 = 0$ or $4\cos^2 x - 4\cos x + 1 = 0$

If we substitute $w = \cos x$, we get

$4w^2 - 4w + 1 = 0 \iff (2w-1)^2 = 0$

so $w = \frac{1}{2}$ or $\cos x = \frac{1}{2}$

so the solutions in $[0, 2\pi)$ are $\boxed{x = \frac{\pi}{3}}$

or $\boxed{x = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}}$

"+" since 15° is in 1^{st} quadrant

"+" since $\frac{7\pi}{12}$ is in 2^{nd} quadrant

8. Choose ONE of the (A) or (B) below. Only one will be graded.

(A) (8 pts) Use half-angle formulas to find the exact value of each of the following:

(a) $\cos(15^\circ) = +\sqrt{\frac{1+\cos(30^\circ)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

(b) $\sin\left(\frac{7\pi}{12}\right) = +\sqrt{\frac{1-\cos\left(\frac{7\pi}{6}\right)}{2}} = \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} = \sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}} = \frac{\sqrt{2+\sqrt{3}}}{2}$

(B) (8 pts) Find the exact value of $\sin\left(\cos^{-1}(-4/5) + \sin^{-1}(+5/13)\right)$.

Let $\alpha = \cos^{-1}\left(-\frac{4}{5}\right) \Rightarrow \cos\alpha = -\frac{4}{5} \Rightarrow \alpha \in [\frac{\pi}{2}, \pi]$ and $\sin\alpha = \sqrt{1-\left(\frac{4}{5}\right)^2} = \frac{3}{5}$

$\beta = \sin^{-1}\left(+\frac{5}{13}\right) \Rightarrow \sin\beta = +\frac{5}{13} \Rightarrow \beta \in [0, \frac{\pi}{2}]$ and $\cos\beta = \sqrt{1-\left(\frac{5}{13}\right)^2} = \frac{12}{13}$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta =$$

$$= \frac{3}{5}\cdot\frac{12}{13} + \left(-\frac{4}{5}\right)\cdot\frac{5}{13} = \frac{36-20}{65} = \frac{16}{65}$$

9. Choose ONE. Only one will be graded. Note the different point values.

(A) (12 pts) Use the Euler formula to obtain the sum formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

(B) (8 pts) Use the sum formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$ to obtain the double angle formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

(A) Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$

So $e^{i(\alpha+\beta)} = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$. (1)

But also $e^{i(\alpha+\beta)} = e^{i\alpha + i\beta} = e^{i\alpha}\cdot e^{i\beta} = (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$

$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$ (2)

From (1) & (2) we get

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

(B) If in the above sum formulas, we make $\alpha=\beta=\theta$, we get

$\cos(2\theta) = (\cos\theta)(\cos\theta) - (\sin\theta)(\sin\theta) = \cos^2\theta - \sin^2\theta$

and $\sin(2\theta) = 2\sin\theta\cos\theta$

Using Pythagorean identity, the formula for $\cos(2\theta)$ also can be rewritten as

$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$