



Asset pricing with incomplete information and fat tails

Prasad V. Bidarkota^{a,*}, Brice V. Dupoyet^b, J. Huston McCulloch^c

^a Department of Economics, University Park DM 320A, Florida International University, Miami, FL 33199, USA

^b Department of Finance, CBA, University Park RB 209A, Florida International University, Miami, FL 33199, USA

^c Department of Economics, 410 Arps Hall, 1945 N. High Street, Columbus, OH 43210, USA

ARTICLE INFO

Article history:

Received 21 May 2007

Accepted 23 January 2009

Available online 5 February 2009

JEL classification:

G12

G13

E43

Keywords:

Asset pricing

Incomplete information

Time-varying volatility

Fat tails

Stable distributions

ABSTRACT

We study a consumption-based asset pricing model with incomplete information and α -stable shocks. Incomplete information leads to a non-Gaussian filtering problem. Bayesian updating generates fluctuating confidence in the agents' estimate of the persistent component of the dividends' growth rate. This has the potential to generate time variation in the volatility of model-implied returns, without relying on discrete shifts in the drift rate of dividend growth rates. A test of the model using US consumption data shows that implied returns display significant volatility persistence of a magnitude comparable to that in the data.

© 2009 Elsevier B.V. All rights reserved.

1. Introduction

Models of asset pricing featuring incomplete information have been used successfully to rationalize several key stylized facts of financial returns (David, 1997; Veronesi, 1999, 2004). The success of incomplete information models derives in part from time varying dispersion in the posterior density of the unobserved drift rate, a feature alternatively referred to as fluctuating confidence in David (1997) and as Bayesian uncertainty in Veronesi (2004). Time varying dispersion in the filter or posterior density arises in these studies because the dividend drift is assumed to be subject to discrete breaks, governed either by a Poisson arrival or Markov switching process.

In the absence of such discrete breaks, the incomplete information setting in these studies reduces to a standard linear Gaussian state space framework for which the Bayes' updating rule is given by the classical Kalman filter. It is well known that in such settings, with a Gaussian prior, the Kalman filter density has a constant dispersion (after transients have died out from startup of the filter) and specifically does not react to the arrival of new information on dividends each period. Asset pricing and portfolio allocation problems in such a classical setting have been studied in Dothan and Feldman (1986), Detemple (1986), Gennotte (1986), and more recently, in Brennan and Xia (2001).

A non-constant dispersion in the filter density can also be obtained without resorting to discrete breaks or non-Gaussian priors as in Detemple (1991). Any non-Gaussian and/or nonlinear filtering setup would deliver this. Kitagawa (1987) proposes using linear state space models with heavy-tailed errors for capturing shifts in the mean and/or variance of a time series. One notable advantage of these models over discrete switching models is that they can capture a continuum of regime changes.

* Corresponding author. Tel.: +1 305 348 6362; fax: +1 305 348 1524.

E-mail addresses: bidarkot@fiu.edu (P.V. Bidarkota), dupoyetb@fiu.edu (B.V. Dupoyet), mcculloch.2@osu.edu (J.H. McCulloch).

In this paper, we set out to study asset pricing with incomplete information in a linear non-Gaussian setting where the exogenous dividends process is not subject to discrete breaks or regime shifts but is instead driven by stochastic shocks with continuous fat tailed probability distributions. The point of departure for our study from the extant literature requires an appropriate continuous probability distribution with fat tails. One immediately runs into difficulties here because, as Geweke (2001) notes, the theory of choice under uncertainty in such settings often breaks down under the constant relative risk-aversion (CRRA) utility function. Geweke (2001) specifically demonstrates the failure of the choice theory with Student-*t* distributions in such circumstances.

The family of α -stable distributions provides a way out of this difficulty. These distributions have a fairly long history in finance, going back to early work by Mandelbrot (1963). A comprehensive survey on the financial applications of these distributions is provided by McCulloch (1996). The $\alpha \in (0, 2]$ parameter (along with three other parameters including a skewness parameter $\beta \in [-1, 1]$) characterizes these distributions, with $\alpha = 2$ resulting in the Gaussian distribution and $\alpha < 2$ resulting in fat-tailed distributions. While the difficulty noted by Geweke (2001) also applies to the general family of α -stable distributions with arbitrary skewness β , the sub-family of these distributions with maximal negative skewness $\beta = -1$ provides an operational theory of choice under uncertainty.

In recent work, Carr and Wu (2003) use this sub-family of α -stable distributions with maximal negative skewness for capturing the observed behavior of the volatility smirk implied by S&P 500 option prices. They too are forced to work with these distributions by imposing $\beta = -1$ in order to ensure finiteness of call option values.

Consequently, in this paper we study asset pricing with incomplete information in a setting where the observed dividend growth rate is modeled as the sum of an unobservable persistent component, assumed to be an autoregressive (AR) process driven by α -stable shocks with maximal negative skewness, and noise. An incomplete information Gaussian asset pricing model is a special case. We characterize the solution to the asset pricing model in such a setting. We then calibrate the model to data on quarterly US per capita consumption, and study the equilibrium implications of the time varying dispersion in the filter density on the characteristics of implied returns. Specifically, we examine the ability of the model to reproduce some key stylized facts of observed returns, including the magnitudes of their key moments, time-varying volatility, and predictability.

The paper is organized as follows. We describe the economic environment and the asset pricing model in Section 2. We study the solution to the model in Section 3. We tackle empirical issues including estimation of the model in Section 4. We analyze the model implications, including those on implied rates of return, in Section 5. The last section provides some conclusions derived from the paper.

2. The asset pricing model

In this section we lay out the economic environment, including specification of exogenous stochastic processes and information structure in the model.

2.1. Pure exchange economy

In a single good Lucas (1978) economy, with a representative utility-maximizing agent and a single asset that pays exogenous dividends of non-storable consumption goods, the first-order Euler condition is

$$P_t U'(C_t) = \theta E_t U'(C_{t+1}) [P_{t+1} + D_{t+1}]. \tag{1}$$

Here, P_t is the real price of the single asset in terms of the consumption good, $U'(C)$ is the marginal utility of consumption C for the representative agent, θ is a constant subjective discount factor, D is the dividend from the single productive unit, and E_t is the mathematical expectation, conditioned on information available at time t .

Assume a CRRA utility function with risk-aversion coefficient γ

$$U(C) = (1 - \gamma)^{-1} C^{1-\gamma}, \quad \gamma \geq 0. \tag{2}$$

Let v_t denote the price-dividend ratio, i.e. $v_t = P_t/D_t$. Since consumption simply equals dividends in this simple model, i.e. $C = D$ every period, Eq. (1) reduces to

$$v_t = E_t \theta (D_{t+1}/D_t)^{1-\gamma} [v_{t+1} + 1]. \tag{3}$$

Thus, this equation implicitly defines the solution to the asset pricing problem in this model. One specifies an exogenous stochastic process for dividends and solves for the price dividend ratio v_t .

2.2. Simplifying the difference equation for the P/D ratios

Let $x_t = \ln(D_t/D_{t-1})$ denote the natural logarithm of the dividend growth rate. Defining $m_{t+1} \equiv \theta \exp[(1 - \gamma)x_{t+1}]$, we can rewrite Eq. (3) as

$$v_t = E_t m_{t+1} [v_{t+1} + 1]. \tag{4}$$

On forward iteration and imposing the transversality condition, this equation yields:

$$v_t = \sum_{i=1}^{\infty} \left(E_t \prod_{j=1}^i m_{t+j} \right). \quad (5)$$

Thus, the solution to the price–dividend ratio can be found by evaluating the conditional expectations on the right hand-side of Eq. (5), under a specified exogenous stochastic process for the dividend growth rates.

2.3. Specification of the endowment process

We assume that dividend growth rates stochastically evolve according to the following process:

$$x_t = \mu_t + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } N(0, \sigma^2), \quad (6a)$$

$$\mu_t - \bar{\mu} = \rho(\mu_{t-1} - \bar{\mu}) + \eta_t, \quad 0 \leq \rho < 1, \quad \eta_t \sim \text{iid } S(\alpha, \beta, c, 0). \quad (6b)$$

We assume that ε_t and η_t are independent of each other contemporaneously as well as at all leads and lags. Here, $S(\alpha, \beta, c, 0)$ represents a stable distribution with characteristic exponent $\alpha \in (0, 2]$, skewness parameter β , scale parameter c , and location parameter set to 0. McCulloch (1996) provides a review of these distributions and lists some of their key properties.

Based on these properties, the characteristic exponent α is restricted to be greater than one so that the innovation term η_t has a well-defined mean, equal to the location parameter, assumed zero in Eq. (6b). From an empirical viewpoint, this restriction is completely innocuous, since for most economic time series α is often estimated to be much higher. For technical reasons that will be evident in Section 3, we need the AR coefficient ρ in Eq. (6b) to be non-negative. This is not a limitation of the model from an empirical viewpoint since, as we shall see in Section 4.3, ρ is estimated to be positive and large.

We also study a benchmark case where, in Eq. (6b), $\eta_t \sim \text{iid } N(0, \sigma_\eta^2)$.¹ From the properties of stable distributions, this is obtained by setting $\alpha = 2$ in the process given in Eq. (6b). In this case, β loses its effect on the distribution of η_t and is unidentified, and $\sigma_\eta^2 = 2c^2$.

2.4. Incomplete information structure of the economy

We assume that agents in the economy have full knowledge about the structure of the economy. They know the stochastic process governing the evolution of the dividend growth rates, including the parameters of the process. They observe the dividend stream (and hence the realized dividend growth rates x_t as well). However, we assume that agents do not ever observe the persistent component μ_t of the dividend growth rates.

Agents need to form conditional expectations of μ_t in order to compute the expected future dividend payouts, and hence determine equilibrium prices. We assume that agents form conditional expectations on μ_t based on Bayesian updating rules. The filtering problem facing the agents is complicated here by the assumption of a non-Gaussian distribution for η_t .

In the benchmark case where η_t is Gaussian, agents face a linear Gaussian filtering problem. In this case, the conditional density of μ_t is Gaussian (see, for instance, Harvey, 1992, Chapter 3) and, therefore, completely specified by its conditional mean and variance. These are given recursively by the classic Kalman filter. When η_t is non-Gaussian, Bayesian updating results in a nonlinear globally optimal filter that leads to a recursive form for the conditional probability density $p(\mu_t | x_1, x_2, \dots, x_t)$.

2.5. Benchmark case-complete information

In a benchmark full information economy, we assume that the innovation ε_t in Eq. (6a) has zero variance (i.e. ε_t is trivially zero). In this case, $\mu_t = x_t$, and therefore agents actually observe μ_t . This model is studied in Bidarkota and McCulloch (2003). A Gaussian version of such a full information model where $\eta_t \sim \text{iid } N(0, \sigma_\eta^2)$ is studied in Burnside (1998).

3. Solution to the model

We now proceed to evaluate Eq. (5) for the price–dividend ratio, and study some of its properties, under the assumed process for the dividend growth rates given in Eqs. (6).

¹ The model in Eqs. (6) has a reduced form ARMA(1,1) representation. Bansal and Yaron (2004) study such a model with conditionally heteroskedastic Gaussian errors with non-expected recursive utility specification of Epstein and Zin (1989). Note that in their setup, unlike ours as will become evident in the next subsection, there is complete information.

3.1. Finiteness of conditional expectations

Theory of choice under uncertainty with CRRA utility and fat-tailed distributions is extremely fragile (Geweke, 2001). The difficulty stems from the non-existence of the conditional expectation in Eq. (5) for most common leptokurtic distributions, including the Student-*t*. Under the assumed distribution $\eta_t \sim iid S(\alpha, \beta, c, 0)$ in the dividend growth rate process in Eq. (6b), Appendix A shows that the conditional expectation in Eq. (5) is finite only when $\beta \text{sign}(1 - \gamma) = -1$. This condition is satisfied if either: (i) $\beta = -1$ and $\gamma < 1$ or (ii) $\beta = +1$ and $\gamma > 1$. Thus, an operational theory of choice under uncertainty can be worked out under these conditions.

One can obtain an operational theory of choice under uncertainty without having to impose the condition $\beta \text{sign}(1 - \gamma) = -1$ by dampening the α -stable distributions, as in the Dampened Power Law process of Wu (2006) and Bidarkota and Dupoyet (2007), or by truncating them. However, in that case, one loses the convenience of solving the asset pricing model in a simple way as will become evident in the next subsection. Moreover, in our view, imposing $\beta \text{sign}(1 - \gamma) = -1$ for the purpose of demonstrating that filtering in leptokurtic non-Gaussian settings would generate volatility persistence is not necessarily severe.

In the remainder of the paper, we assume that $\beta = -1$ and $\gamma < 1$. This choice for β is consistent with negative skewness in dividends growth data and with evidence on $\gamma < 1$ in Arrow (1971), Hansen and Singleton (1982), and Epstein and Zin (1991).² Carr and Wu (2003) refer to the $\beta = -1$ case as the finite moment log-stable process.

3.2. Solution for the P/D ratios

Under the assumptions $\beta = -1$ and $\gamma < 1$, one can derive a tractable expression for the price–dividend ratio v_t in Eq. (5). Appendix B shows that v_t can be reduced to

$$v_t = \sum_{i=1}^{\infty} \theta^i [E_t \exp\{b_i(\mu_t - \bar{\mu})\}] \exp \left[i\bar{\mu}(1 - \gamma) + i(1 - \gamma)^2 \frac{\sigma^2}{2} + \left\{ - \left[\left(\frac{1 - \gamma}{1 - \rho} \right)^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right) \right] \sum_{j=1}^i (1 - \rho^j)^\alpha \right\} \right], \quad (7)$$

where $b_i \equiv (1 - \gamma)(\rho/1 - \rho)(1 - \rho^i) > 0$. It is not possible to evaluate the conditional expectation term in the above equation analytically under the assumed process for μ_t in Eq. (6b).

3.3. Convergence of the P/D ratios

The following theorem provides conditions for the price–dividend ratio to be finite.

Theorem 1. The series in Eq. (7) converges if and only if

$$r \equiv \theta \exp \left[(1 - \gamma)\bar{\mu} + (1 - \gamma)^2 \frac{\sigma^2}{2} + \left\{ - \left[\left(\frac{1 - \gamma}{1 - \rho} \right)^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right) \right] \right\} \right] < 1. \quad (8)$$

Proof. Proof follows along the lines of the arguments in Appendix D of Bidarkota and McCulloch (2003).

Finiteness of the price–dividend ratio ensures that the expected discounted utility is finite in this model (see Burnside 1998). The next theorem derives an expression for the mean of the price–dividend ratio and provides conditions under which this mean is finite.

Theorem 2. The mean of the price dividend ratio is given by

$$E(v_t) = \sum_{i=1}^{\infty} \theta^i \exp \left[i\bar{\mu}(1 - \gamma) + i(1 - \gamma)^2 \frac{\sigma^2}{2} + \left\{ - \left[b_i^\alpha \frac{c^\alpha}{1 - \rho^\alpha} \right] \sec\left(\frac{\pi\alpha}{2}\right) \right\} + \left\{ - \left[\left(\frac{1 - \gamma}{1 - \rho} \right)^\alpha c^\alpha \sec\left(\frac{\pi\alpha}{2}\right) \right] \sum_{j=1}^i (1 - \rho^j)^\alpha \right\} \right]. \quad (9)$$

It is finite if $r < 1$, where r is the constant defined in Theorem 1.

Proof. Proof follows along the lines of the arguments in Appendix E of Bidarkota and McCulloch (2003).

² Although the asset pricing theory with CRRA utility and stable distributions would go through with $\beta = +1$ and $\gamma > 1$, we strongly suspect that this would lead to mean implied equity returns, discussed in Section 5 below, being infinite. It is easy to prove this analytically in the complete information case, with the autoregressive coefficient ρ in Eq. (6b) set to zero. However, it is not possible to demonstrate this analytically with the more general process for dividend growth rates that we have in Eqs. (6) in the paper. This is one other reason why we do not entertain this alternative possibility in the paper.

3.4. Solution under Gaussian distribution for η_t

In the benchmark case when $\eta_t \sim iid N(0, \sigma_\eta^2)$, we can obtain all the results derived in the previous subsections simply by setting $\alpha = 2$. In this case, the skewness parameter of the stable distributions β loses its significance. We no longer need the restriction $\beta \text{sign}(1 - \gamma) = -1$ in order to ensure finiteness of the conditional expectations term in Eq. (5), and hence of the price–dividend ratio itself and its mean value in Eqs. (7) and (9), respectively.

As discussed in Section 2.4, in the Gaussian case, the conditional density of μ_t is Gaussian, and its conditional mean and variance are given by the Kalman recursions. In this case, the conditional expectations term $E_t \exp\{b_i(\mu_t - \bar{\mu})\}$ appearing in the formula for the price–dividend ratio given in Eq. (7) can be evaluated using the formula for the moment generating function of Gaussian random variables.

3.5. Solution under complete information

In the complete information benchmark case, recall from Section 2.5 that $\sigma^2 = 0$ and $\mu_t = x_t$, which is observed at time t . All the analysis of Section 3 goes through exactly as in the incomplete information case. The price–dividend ratio and its related properties in the benchmark complete information model with stable distribution for η_t are derived in Bidarkota and McCulloch (2003) and with Gaussian distribution for η_t in Burnside (1998).

4. Empirical estimation of the model

In this section we report the data used to calibrate the theoretical model of Section 2, discuss maximum likelihood estimation of the dividend growth rates process given in Eqs. (6), and report estimates of the parameters obtained.

4.1. Data issues

We calibrate the asset pricing model to quarterly real per capita US consumption growth rates on non-durables and services from 1952:1 to 2004:2. Nominal seasonally adjusted per capita consumption data obtained from NIPA tables are deflated using the CPI index. Summary statistics indicate an annualized mean growth rate of 2.02% and a standard deviation of 1.34%. Skewness is estimated to be -0.40 and statistically different from 0 at the 1% level, indicating significant negative skewness. This provides justification for our choice of a negative value for β in Section 3.1. Kurtosis is estimated to be 4.29 and statistically different from 3 at the 1% level. This provides preliminary justification for our use of fat tailed distribution in the specification of Eq. (6b). The Jarque–Bera test easily rejects normality at better than the 1% level. The first order autocorrelation coefficient is 0.18 and statistically different from 0 at the 1% level. This provides preliminary empirical justification for our restriction $\rho \geq 0$ in the specification of Eq. (6b).

In order to assess the suitability of an α -stable process for consumption growth rates, we estimated the tail index of their density (Kearns and Pagan, 1997) using the Hill estimator (1975). Estimation results with monthly consumption data (not reported) indicate that the density tail index does not increase monotonically as the data is time aggregated. This is contrary to what one would expect under Gaussian densities. On the other hand, the estimates are not stable with time aggregation either, contrary to what is to be expected for α -stable densities (Wu, 2006). Also, sample estimates of skewness exhibit great instability with time aggregation, supporting the α -stable hypothesis. On the other hand, sample estimates of kurtosis decline with time aggregation, suggesting non- α -stable behavior. Overall, preliminary evidence in favor of the α -stable hypothesis appears to be mixed.

4.2. Maximum likelihood estimation

The dividend growth rates process in Eqs. (6) constitutes a linear non-Gaussian state space model when η_t has the stable distribution. Eq. (6a) is the observation equation and Eq. (6b) is the state transition equation. The non-Gaussian nature of the model renders the Kalman filter suboptimal. Recursive formulae for obtaining the conditional densities of the state variable μ_t , as well as the likelihood function, are available from an algorithm by Sorenson and Alspach (1971).

Let X_t denote the history of dividend growth rates observed at time t , i.e. $X_t \equiv \{x_1, \dots, x_t\}$. The recursive formulae for obtaining one-step ahead predictive and filtering densities, due to Sorenson and Alspach (1971), are as follows:

$$p(\mu_t | X_{t-1}) = \int_{-\infty}^{\infty} p(\mu_t | \mu_{t-1}) p(\mu_{t-1} | X_{t-1}) d\mu_{t-1}, \quad (10a)$$

$$p(\mu_t | X_t) = p(x_t | \mu_t) p(\mu_t | X_{t-1}) / p(x_t | X_{t-1}), \quad (10b)$$

$$p(x_t | X_{t-1}) = \int_{-\infty}^{\infty} p(x_t | \mu_t) p(\mu_t | X_{t-1}) d\mu_t. \quad (10c)$$

Finally, the log-likelihood function is given by

$$\log p(x_1, \dots, x_T) = \sum_{t=1}^T \log p(x_t | X_{t-1}). \tag{11}$$

These formulae have been applied to non-Gaussian data and extended to include a smoother formula by Kitagawa (1987).

In the Gaussian case, these integrals can be evaluated analytically and they collapse to the Kalman recursions. In most other circumstances, the integrals cannot be evaluated in closed form and one has to resort to numerical integration based either on quadrature techniques (Kitagawa, 1987) or Monte Carlo methods (Durbin and Koopman, 2000). In this paper we evaluate the integrals using quadrature methods. Details on the numerical method employed and its accuracy are omitted here for the sake of brevity but are available in a related context in Bidarkota and McCulloch (1998, 2004) and Bidarkota (2001, 2003). Kitagawa (1987) also provides extensive details on accuracy of quadrature techniques for estimation in this context.

The probability density for stable distributions is obtained by Fourier inversion of their characteristic function available as an exact analytical formula (Eqs. (A.2) and (A.3) of Bidarkota and McCulloch, 2003) using the Fast Fourier Transform (FFT) methods discussed in Mittnik et al. (1999).

4.3. Parameter estimates

Maximum likelihood parameter estimates of the consumption growth rate process in Eqs. (6) are reported in Table 1 (Panel A). Three restricted versions of this most general model are also reported in Panels B to D.

Parameter estimates from Panel A indicate a mean consumption growth rate of 0.49% per quarter, or 1.96% per annum. The AR parameter ρ is estimated to be 0.69, somewhat lower than the value of 0.89 reported in Veronesi (2004) with an AR

Table 1
Parameter estimates.

Panel A: Incomplete information, stable model					
$\bar{\mu}$	ρ	α	c	σ	Log likelihood
0.0049	0.6889	1.8564	0.0014	0.0058	755.6100
0.0007	0.1724	0.2500	0.0007	0.0005	
Panel B: Incomplete information, Gaussian model					
$\bar{\mu}$	ρ	α	σ_η	σ	Log likelihood
0.0050	0.7396	2.0000	0.0022	0.0058	755.4003
0.0007	0.1260	(restricted)	0.0008	0.0004	
Panel C: Complete information, stable model					
$\bar{\mu}$	ρ	α	c		Log likelihood
0.0051	0.1343	1.9140	0.0043		755.5771
0.0003	0.0222	0.0545	0.0002		
Panel D: Complete information, Gaussian model					
$\bar{\mu}$	ρ	α	σ_η		Log likelihood
0.0050	0.1758	2.0000	0.0066		753.5202
0.0006	0.0674	(restricted)	0.0003		

This table reports the maximum likelihood estimates of the model for dividend growth rates, $x_t = \mu_t + \varepsilon_t$ where $\varepsilon_t \sim iid N(0, \sigma^2)$ and where the unobserved persistent component μ_t follows:

$$\mu_t - \bar{\mu} = \rho(\mu_{t-1} - \bar{\mu}) + \eta_t \quad \text{with } 0 \leq \rho < 1 \text{ and } \eta_t \sim iid S(\alpha, \beta, c, 0).$$

The model is calibrated to quarterly real per capita US consumption growth rates on non-durables and services from the first quarter of 1952 through the second quarter of 2004. Nominal seasonally adjusted per capita consumption data obtained from NIPA tables are deflated using the CPI index.

Panel A reports estimates for the most general model. Panel B reports estimates for the special case where $\eta_t \sim iid N(0, \sigma_\eta^2)$. Panels C and D report estimates for the complete information counterparts of panels A and B, by setting ε_t to zero.

Conditional densities of the state variable μ_t are obtained by applying the algorithm by Sorenson and Alspach (1971) in panel A and a Kalman filter in panel B. The probability density for stable distributions is obtained by Fourier inversion of their characteristic function available as an exact analytical formula using the Fast Fourier Transform (FFT) methods discussed in Mittnik et al. (1999). Standard errors are reported below each parameter estimate.

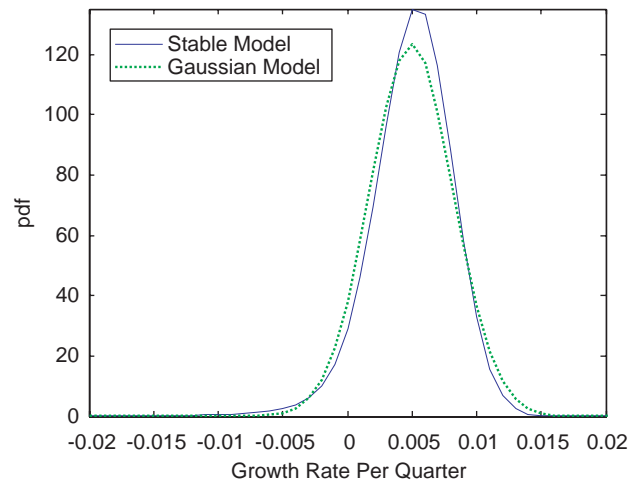


Fig. 1. Unconditional distribution of μ_t . The figure uses the maximum likelihood parameter estimates reported in Panels A and B of Table 1 to plot the unconditional distribution of μ_t , the persistent component of dividend growth rates defined by $x_t = \mu_t + \varepsilon_t$ and $\mu_t - \beta = \rho(\mu_{t-1} - \beta) + \eta_t$, for both the stable model where $\eta_t \sim iid S(\alpha, \beta, c, 0)$ and the Gaussian model where $\eta_t \sim iid N(0, \sigma_\eta^2)$, with $\varepsilon_t \sim iid N(0, \sigma^2)$ and $0 \leq \rho < 1$.

model with asymmetric jumps. Nonetheless, it is statistically significantly different from 0 by the usual t -test at better than the 1% significance level.

Parameter estimates for the incomplete information Gaussian model in Panel B are very similar, and the maximized log-likelihood value only drops slightly in this case. The likelihood ratio (LR) statistic for normality (test for $\alpha = 2$) is 0.42 and is not rejected.^{3,4}

Fig. 1 plots the unconditional distribution of μ_t for both the stable and Gaussian models implied by Eq. (6b), using the maximum likelihood parameter estimates reported above. With the stable index α estimated to be 1.86, so close to the value of 2 for a Gaussian distribution, the differences in the unconditional densities for the stable and Gaussian models are modest. The assumption $\beta = -1$ does not generate much skewness in the unconditional distribution for this high value of α , as evident in the figure, and thus does not seem overly restrictive.

Fig. 2 plots the conditional probability densities $p(\mu_t | x_1, x_2, \dots, x_t)$. Close examination of the plots indicate that the conditional densities in the stable case display varied behavior, being at times even multimodal. Bidarkota and McCulloch (1998) and Bidarkota (2003) provide detailed examination of such densities in non-Gaussian state space models with stable errors fit to inflation data. On the other hand, as noted earlier, the conditional densities in the Gaussian case are Gaussian.

Fig. 3 plots the mean of these filter densities $E(\mu_t | x_1, x_2, \dots, x_t)$, along with the observed consumption growth rates x_t , and Fig. 4 plots their standard deviations. It is clear that the variance of the filter density in the Gaussian case quickly reaches a constant value. On the other hand, the variance of the filter density in the stable case never stabilizes to a constant value but is forever fluctuating. David (1997) refers to this as ‘fluctuating confidence’ in the investors’ estimate of the unobservable component of the dividend growth rate and Veronesi (2004) refers to this as ‘Bayesian uncertainty’.⁵ This fluctuating confidence largely drives the time varying characteristics of returns implied by our asset pricing model.

The complete information stable model parameter estimates are reported in Panel C of Table 1. The AR coefficient ρ is now only 0.13, as against 0.69 for the incomplete information model, but still statistically significantly different from 0 by

³ The distribution of the LR test statistic in this instance is not standard χ^2 because the null hypothesis lies on the admissible boundary of α . The 0.05 level critical value for such a test, available through Monte Carlo simulations from McCulloch (1997, Table 4, panel b), equals 1.12.

⁴ One could alternatively consider a dividends growth rate process with ε_t distributed as $S(\alpha, \beta, c, 0)$ and $\eta_t \sim iid N(0, \sigma_\eta^2)$ in Eqs. (6). We estimated such a model by ML and found that parameter estimates are very similar across the two models but the maximized log-likelihood is actually higher (758.0634, LR test would reject the Gaussian model easily). Given the similarities in parameter estimates, our asset pricing implications are likely to be materially unaffected. We do not entertain this alternative specification, despite its higher log-likelihood, for two reasons. The asset pricing solution with such a specification also requires $\beta = -1$ and $\gamma < 1$. We are reluctant to specify a $S(\alpha, \beta, c, 0)$ distribution with $\beta = -1$ for the measurement equation error in the state space model if one views ε_t as measurement error in the consumption growth rate. A second reason is that we follow Veronesi (2004) as well as the literature on Markov switching processes (e.g., Brandt et al., 2004) in keeping the measurement equation shock Gaussian and the state dynamics non-Gaussian.

⁵ Filtering in general non-Gaussian settings, not just those with α -stable distributions, leads to fluctuating confidence. To verify this, we estimated the model in Eqs. (6) under two alternative fat-tailed distributions for η_t , namely the Extreme Value and the Pearson Type IV distributions. Corresponding plots of the filter densities in these two cases reveal similar behavior of the filter standard deviations as in Fig. 4, indicating that these alternative fat-tailed distributions also generate fluctuating confidence.

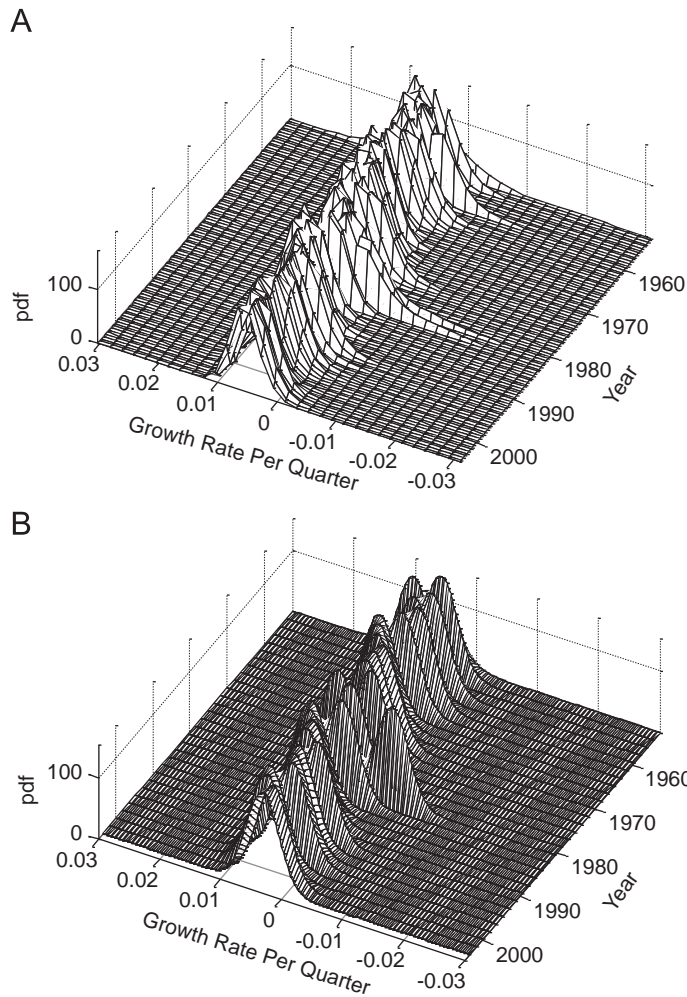


Fig. 2. Filtered probability densities $p(\mu_t|x_1, x_2, \dots, x_t)$. The figure plots the filtered conditional probability densities $p(\mu_t|x_1, x_2, \dots, x_t)$. Panel A plots the densities for the stable case with the densities estimated by using the algorithm by Sorenson and Alspach (1971) and the maximum likelihood parameter estimates of Panel A in Table 1. Panel B plots the densities for the Gaussian case with the densities estimated by using a Kalman filter and the maximum likelihood parameter estimates of Panel B in Table 1.

the usual t -test at better than the 1% significance level. The LR test statistic for complete information versus incomplete information model turns out to be just 0.07, and is not rejected using critical values from the χ^2_1 at usual significance levels.

The complete information Gaussian model parameter estimates are reported in Panel D of Table 1. These estimates change only slightly from those in Panel C. However, the maximized log-likelihood shows a large drop. The LR test statistic for normality (test for $\alpha = 2$) is calculated to be 4.11 and rejected. Thus, with complete information models, there is significant statistical support for stable shocks. The LR test statistic for complete information versus incomplete information model in the Gaussian case turns out to be 3.76, with a χ^2_1 p -value of 0.05. Thus, with Gaussian shocks, there is significant statistical support for the incomplete information model.

5. Analysis of model implications

In this section we discuss the implications of the theoretical model of Section 2 and report on the results obtained in analyses of unconditional and conditional properties of model implied rates of returns from a simulation setup.

5.1. Model-implied rates of return

Equilibrium gross equity returns R_t^e on assets held from periods t to $t+1$ are given by

$$R_t^e = \left(\frac{P_{t+1} + D_{t+1}}{P_t} \right). \tag{12}$$

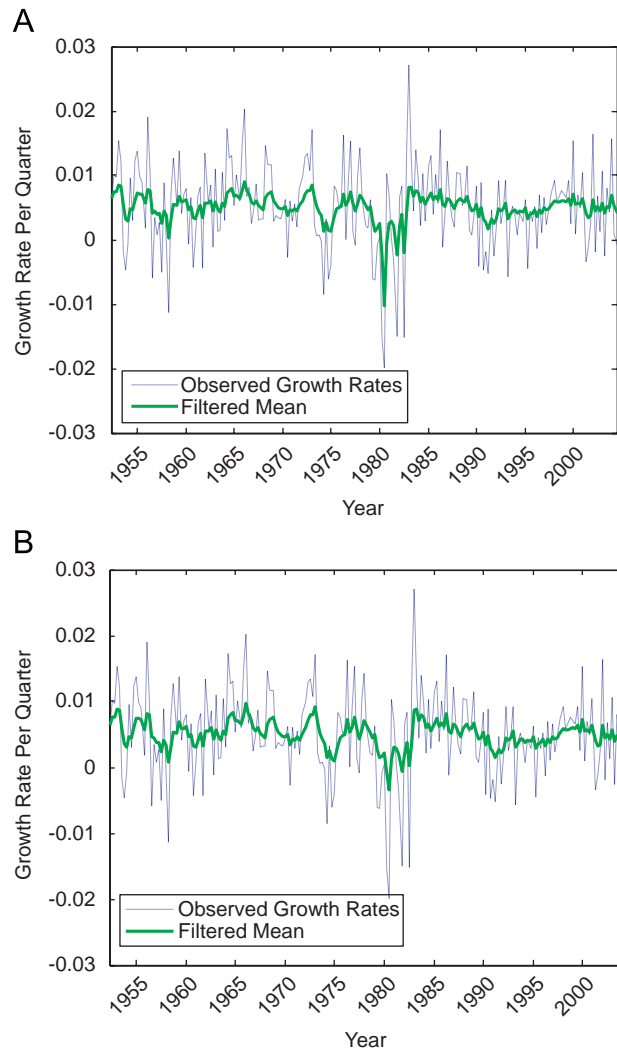


Fig. 3. Filtered mean $E(\mu_t|x_1, x_2, \dots, x_t)$. The figure plots the mean of the filtered densities $E(\mu_t|x_1, x_2, \dots, x_t)$, along with the observed consumption growth rates x_t . Panel A plots the mean of the filtered densities for the stable case with the densities (plotted in Fig. 2A) estimated by using the algorithm by Sorenson and Alspach (1971) and the maximum likelihood parameter estimates of Panel A in Table 1. Panel B plots the mean of the filtered densities for the Gaussian case with the densities (plotted in Fig. 2B) estimated by using a Kalman filter and the maximum likelihood parameter estimates of Panel B in Table 1.

Using $v_t = P_t/D_t$ and $x_t = \ln(D_t/D_{t-1})$, this reduces to

$$R_t^e = \left(\frac{1 + v_{t+1}}{v_t} \right) \exp[x_{t+1}]. \quad (13)$$

It is not possible to analytically evaluate the population mean of the implied equity returns, i.e. $E(R_t^e)$, in our model given the expression for v_t in Eq. (7).

The price of a risk free asset P_t^f in our endowment economy guarantees one unit of the consumption good on maturity. Our assumption $\beta = -1$ implies that P_t^f is infinite and hence the gross risk free returns are zero. Formal proof follows along the lines of the argument in Appendix F of Bidarkota and McCulloch (2003). Presumably, in the model for the dividends process, the uncertainty is so overwhelmingly unfavorable that as long as investors are risk averse, they are willing to pay an infinite amount to guarantee themselves strictly positive consumption next period. Infinite prices for risk free assets simply mean that these assets cannot exist in the economy under the assumed stochastic process for dividends.

One can get around the difficulty of infinite risk-free asset prices by truncating or dampening the α -stable distributions in Eq. (6b), as in the dampened power law process in Wu (2006). Although dampening/truncation may be appealing, we lose the convenience of solving the asset pricing model in a simple way. In any case, our main objective in this paper is to demonstrate that filtering in fat-tailed non-Gaussian continuous-valued stochastic setting can generate volatility

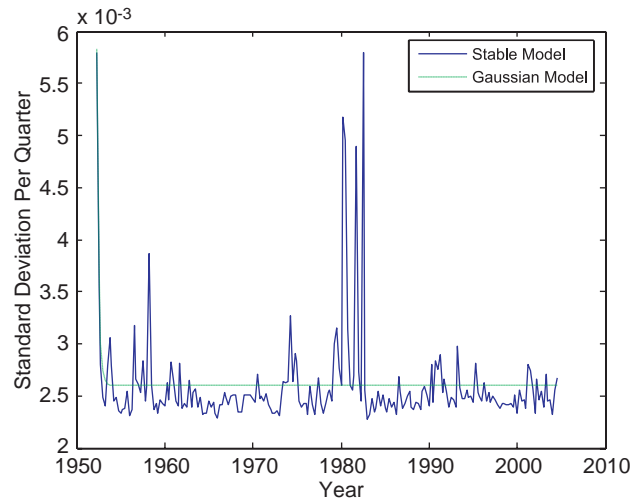


Fig. 4. Filtered standard deviation of $p(\mu_t|x_1, x_2, \dots, x_t)$ -fluctuating confidence. The figure plots the standard deviation of the filtered densities for both the stable and Gaussian incomplete information models. The standard deviations of the filtered densities are estimated by using the algorithm by Sorenson and Alspach (1971) and the maximum likelihood parameter estimates of each model.

persistence in implied returns. Therefore, in what follows, we restrict ourselves to an analysis of implied returns on risky assets.

5.2. Unconditional moments

Table 2, Panel A reports unconditional moments of quarterly value-weighted real returns on all NYSE, AMEX, and NASDAQ stocks obtained from CRSP dataset for the period 1952:1–2004:2. We subtract CPI inflation from nominal returns to obtain real returns, expressed in percent per annum.

Panel A indicates that quarterly real returns have a mean of 8.07% per annum and a standard deviation of 16.77. Panels B to E report the unconditional moments for returns implied by our theoretical model of Section 2, obtained by simulating it at an annual frequency. Moments are reported for various values of the discount factor θ and the risk-aversion coefficient γ .

The maximum implied mean returns from our incomplete information stable model are only 3.90% and the maximum standard deviation is only 1.75%. Overall, it is clear from looking at all the panels that none of the models do a good job of replicating the unconditional moments of equity returns. This is simply a manifestation of the equity premium puzzle of Mehra and Prescott (1985).

It is clear from an examination of all the panels that adding incomplete information to the asset pricing model reduces the mean and raises the standard deviation of implied equity returns slightly. Generalizing from Gaussian to stable models lowers both the mean and the standard deviation of implied equity returns in the incomplete information case but raises these moments in the complete information case.

The benchmark complete information asset pricing model, with stable and Gaussian dividend growth rate processes, was studied in Bidarkota and McCulloch (2003). Panels D and E of Table 2 here replicate results reported in that study closely, although the model in that study was calibrated to a random walk process with drift, fit to annual US consumption data for the period 1890–1987. Our results in Table 2 thus indicate that adding incomplete information to the non-Gaussian framework of that study does not generate high enough mean equity returns to conform closely enough to the numbers in the data.

5.3. Time-varying volatility

We now proceed to an evaluation of the conditional moments of implied returns in a simulation setup for $\theta = 0.98$ and $\gamma = 0.9$, since for these preference parameter values the unconditional mean stock returns implied by the incomplete information stable model are closest to their sample counterpart.

Let $r_t \equiv R_t^e - 1$ denote the net rates of return on risky assets. We set up the following model for analyzing net returns:

$$r_t = a_0 + a_5 r_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \sigma_t z_t, \quad z_t \sim iid N(0, 1). \tag{14a}$$

A simple AR(1) process for returns allows for any possible predictability, and has been used by Malik et al. (2005) and Tsiakas (2006).

Table 2
Unconditional moments of returns.

Panel A: Data (1952:1 to 2004:2)					
		$E(R)$	$\sigma(R)$		
Real returns		8.07	16.77		
Panel B: Incomplete information, stable model					
θ	γ	$E(R)$	$\sigma(R)$	$E(V_t)$	
0.98	0.60	3.2901	1.7459	80.599	
0.98	0.75	3.5943	1.6466	64.898	
0.98	0.90	3.8986	1.555	54.321	
0.99	0.60	2.2753	1.7388	406.67	
0.99	0.75	2.5511	1.6363	192.95	
0.99	0.90	2.8494	1.5417	123.08	
Panel C: Incomplete information, Gaussian model					
θ	γ	$E(R)$	$\sigma(R)$	$E(V_t)$	
0.98	0.60	3.31	2.06	81.26	
0.98	0.75	3.62	1.94	65.17	
0.98	0.90	3.93	1.83	54.40	
0.99	0.60	2.30	2.05	420.35	
0.99	0.75	2.58	1.93	195.28	
0.99	0.90	2.88	1.82	123.48	
Panel D: Complete information, stable model					
θ	γ	$E(R)$	$\sigma(R)$	$E(V_t)$	
0.98	0.60	3.34	1.62	82.89	
0.98	0.75	3.65	1.60	65.83	
0.98	0.90	3.97	1.57	54.58	
0.99	0.60	2.33	1.61	456.90	
0.99	0.75	2.61	1.58	201.12	
0.99	0.90	2.92	1.56	124.43	
Panel E: Complete information, Gaussian model					
θ	γ	$E(R)$	$\sigma(R)$	$E(V_t)$	
0.98	0.60	3.31	1.48	82.07	
0.98	0.75	3.63	1.45	65.51	
0.98	0.90	3.94	1.42	54.50	
0.99	0.60	2.30	1.47	438.09	
0.99	0.75	2.58	1.44	198.24	
0.99	0.90	2.89	1.41	123.97	

Panel A reports unconditional moments of quarterly value-weighted real returns on all NYSE, AMEX, and NASDAQ stocks obtained from CRSP data set for the period going from the first quarter of 1952 through the second quarter of 2004. CPI inflation is subtracted from nominal returns to obtain real returns, expressed in percent per annum. Moments are reported for a range of values for the subjective discount factor θ , and the risk-aversion coefficient γ . Panels B–E report the unconditional moments of simulated returns obtained from the asset pricing model by feeding simulated consumption growth rates data using the estimated parameters from each panel of Table 1. The statistics reported in percentage per annum are the mean returns $E(R)$, standard deviation of returns $\sigma(R)$, and the mean price-dividend ratio $E(V_t)$.

Time-varying volatility of returns σ_t is modeled alternatively as a GARCH(1,1) and an asymmetric GARCH(1,1) process (AGARCH) as follows:

$$\text{GARCH}(1, 1): \sigma_t^2 = a_1 + a_2\sigma_{t-1}^2 + a_3|r_{t-1} - a_0 - a_5r_{t-2}|^2, \quad (14b)$$

$$\text{AGARCH}(1, 1): \sigma_t^2 = a_1 + a_2\sigma_{t-1}^2 + a_3|r_{t-1} - a_0 - a_5r_{t-2}|^2 + a_4I_{t-1}|(r_{t-1} - a_0 - a_5r_{t-2})/\sigma_{t-1}|^2, \quad (14c)$$

where

$$I_{t-1} = \begin{cases} 1 & \text{if } r_{t-1} - a_0 - a_5r_{t-2} < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Table 3
Time-varying volatility of returns.

Panel A: Date (1952:1–2004:2)						
	a_1				a_5	Log likelihood
Homoskedastic case	1119.6475				0.0626	–1030.2289
	110.2488				0.0300	
	a_1	a_2	a_3		a_5	Log likelihood
GARCH(1,1)	313.5011	0.5651	0.1613		0.0954	–1026.9285
	147.2470	0.1646	0.0825		0.0715	
	a_1	a_2	a_3	a_4	a_5	Log likelihood
AGARCH(1,1)	430.9533	0.4169	0.0000	435.3126	0.1390	–1023.6925
	172.0353	0.1683	0.1056	237.1131	0.0767	
Panel B: Incomplete information, Stable model						
	a_1				a_5	Log likelihood
Homoskedastic case	2.2889				0.2321	–7309.8851
	0.0510				0.0154	
	a_1	a_2	a_3		a_5	Log likelihood
GARCH(1,1)	1.4174	0.3057	0.0745		0.1843	–7278.9910
	0.1248	0.1248	0.0239		0.0183	
	a_1	a_2	a_3	a_4	a_5	Log likelihood
AGARCH(1,1)	1.3176	0.3486	0.0000	0.3295	0.1782	–7265.5310
	1.8868	1.2550	0.6308	0.5123	0.0329	
Panel C: Incomplete information, Gaussian model						
	a_1				a_5	Log likelihood
Homoskedastic case	3.7116				0.1838	–8273.7786
	0.0832				0.0171	
	a_1	a_2	a_3		a_5	Log likelihood
GARCH(1,1)	2.9426	0.2073	0.0000		0.1839	–8274.0889
	3.7463	0.1763	0.9198		0.0669	
	a_1	a_2	a_3	a_4	a_5	Log likelihood
AGARCH(1,1)	3.6919	0.0000	0.0000	0.0412	0.1849	–8273.9457
	3.3757	0.9853	0.9633	3.0558	0.0325	
Panel D: Complete information, stable model						
	a_1				a_5	Log likelihood
Homoskedastic case	2.4370				0.1284	–7434.9947
	0.0661				0.0161	
	a_1	a_2	a_3		a_5	Log likelihood
GARCH(1,1)	1.9300	0.2083	0.0000		0.1284	–7435.4662

Table 3 (continued)

	0.2726	0.1034	0.0024		0.0148	
	a_1	a_2	a_3	a_4	a_5	Log likelihood
AGARCH(1,1)	2.4368 1.1883	0.0004 0.0011	0.0000 0.0372	0.0000 0.0436	0.1284 0.0682	-7435.4662
Panel E: Complete information, Gaussian model						
	a_1				a_5	Log likelihood
Homoskedastic case	1.9397 0.0451				0.1963 0.0158	-6979.7742
	a_1	a_2	a_3	a_4	a_5	Log likelihood
GARCH(1,1)	1.8424 0.4957	0.0503 0.5505	0.0000 0.8168		0.1964 0.0676	-6980.0727
AGARCH(1,1)	1.9268 2.0004	0.0000 1.0000	0.0000 0.9074	0.0265 1.5912	0.1971 0.0237	-6979.8422

Panel A reports the estimates of a homoskedastic model, GARCH(1,1) model and AGARCH(1,1) model fitted to quarterly value-weighted real returns on all NYSE, AMEX, and NASDAQ stocks obtained from CRSP data set for the period going from the first quarter of 1952 through the second quarter of 2004. The models assume that returns are conditionally normally distributed. Thus

$$r_t = a_0 + a_5 r_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim \sigma_t z_t, \quad z_t \sim iid N(0, 1)$$

with volatility given by, respectively:

$$\text{Homoskedastic: } \sigma_t^2 = a_1$$

$$\text{GARCH}(1, 1): \sigma_t^2 = a_1 + a_2 \sigma_{t-1}^2 + a_3 |r_{t-1} - a_0 - a_5 r_{t-2}|^2$$

$$\text{AGARCH}(1, 1): \sigma_t^2 = a_1 + a_2 \sigma_{t-1}^2 + a_3 |r_{t-1} - a_0 - a_5 r_{t-2}|^2 + a_4 I_{t-1} |(r_{t-1} - a_0 - a_5 r_{t-2}) / \sigma_{t-1}|^2$$

where

$$I_{t-1} = \begin{cases} 1 & \text{if } r_{t-1} - a_0 - a_5 r_{t-2} < 0 \\ 0 & \text{otherwise} \end{cases}$$

The restrictions $a_1 > 0$, $a_2 \geq 0$, $a_3 \geq 0$, $a_4 \geq 0$, and $|a_5| < 1$ are enforced in all models. Panels B–E report estimates of the above models with simulated returns obtained from the asset pricing model by feeding simulated consumption growth rates data using the estimated parameters from each panel of Table 1. A subjective discount factor θ of 0.98 and a risk-aversion coefficient γ of 0.9 are used to obtain simulated returns since the unconditional mean stock returns implied by the incomplete information stable model are closest to their sample counterpart for these preference parameter values. Standard errors are reported below each volatility parameter estimate.

We restrict $a_1 > 0$, $a_2 \geq 0$, $a_3 \geq 0$, $a_4 \geq 0$, and $|a_5| < 1$. The AGARCH(1,1) process (asymmetric GARCH) allows for leverage effects, captured by the threshold term involving the dummy variable I_{t-1} . Leverage effects indicate that negative shocks to returns have greater effect on future volatility than do positive shocks of equal magnitude.

Table 3, Panel A reports estimates obtained by fitting the above volatility processes to quarterly value-weighted real returns on all NYSE, AMEX, and NASDAQ stocks obtained from CRSP dataset for the period 1952:1–2004:2. We subtract CPI inflation from nominal returns to obtain real returns, expressed in percent per annum for estimation. Panel A indicates that, with estimates for the AGARCH process, quarterly real returns have a volatility persistence parameter a_2 of 0.42. It is well known in the literature that quarterly returns exhibit lower volatility persistence compared to higher frequency returns, such as monthly or weekly returns. The ARCH parameter a_3 is estimated to be 0.00. The leverage parameter a_4 is estimated to be 435.31.

An LR test for homoskedasticity (test for $a_2 = a_3 = a_4 = 0$) rejects easily using the χ^2_3 distribution (p -value less than 0.01). LR test for GARCH versus the AGARCH model (test for $a_4 = 0$) also rejects easily using the χ^2_1 distribution (p -value less

than 0.05), indicating strong statistical significance of the leverage effect. Strictly speaking, the LR test does not have the standard χ^2 distribution for these tests because the null hypotheses lie on the boundary of admissible values for a_2 , a_3 , and a_4 (as also in footnote 2). Andrews (2001) provides recent theoretical advances in this regard. Monte Carlo critical values for these tests generated using 1000 replications indicated even stronger rejection of the two hypotheses.

Panels B to E of Table 3 report maximum likelihood estimates obtained by fitting the above volatility processes to returns implied by the four versions of our models. Estimates of the AGARCH process fit to implied returns from the incomplete information stable model indicate a volatility persistence coefficient of 0.35, an ARCH coefficient of 0.00, and a leverage parameter of 0.33. The low value for the leverage parameter as compared to that for the data is simply a reflection of the fact that our implied returns do not match the unconditional variance of returns in the data very well, as evident from the discussion of unconditional moments in Section 5.3. The overall volatility persistence (a_2+a_3) is 0.35 in implied returns as opposed to 0.42 in the data. LR test for homoskedasticity is strongly rejected in favor of time-varying volatility, and leverage effects are also strongly statistically significant. Our choice of negative skewness in consumption growth rates in Section 3.1 leads to a greater probability of large negative shocks than large positive shocks. This feature of our framework is directly responsible for generating the apparent leverage effects.

Results reported in Panel C indicate that the incomplete information Gaussian model does not generate statistically significant volatility persistence, with the point estimate of the volatility persistence from the AGARCH model at 0.00. An examination of results reported in the remaining panels indicates that implied returns from the complete information models, both stable and Gaussian, completely fail to generate statistically significant volatility persistence and/or the leverage effects.

Implied returns from our incomplete (and complete) information stable model are unlikely to be Gaussian. Therefore, the models for net returns in Eqs. (14) are likely to be misspecified, at least in the stable cases. To account for this, we also estimated versions of the GARCH(1,1) and AGARCH(1,1) models, with the scaled innovation z_t in Eq. (14a) distributed as $iidS(\alpha', \beta', 1, 0)$. The volatility process in Eqs. (14b) and (14c) is formulated in this case in terms of time-varying scale parameter c_t using α -powers instead of squares, as in Bidarkota and McCulloch (2004). Maximum likelihood estimation of this model, and a subsequent LR test for $\alpha = 2$ easily rejects as suspected, indicating statistically significant non-normality of implied returns. An LR test indicates statistically significant volatility persistence with a larger volatility persistence parameter. However, leverage effects are no longer statistically significant.

5.4. Predictability

We go on to analyze the ability of our incomplete information stable model to reproduce other stylized facts about observed returns. Sample autocorrelation coefficients for quarterly value-weighted real returns on the NYSE, AMEX, and NASDAQ stocks analyzed in the previous subsection are 0.06, -0.04 , and -0.02 at the first three lags. The sample autocorrelation coefficients for returns implied by our incomplete information stable model are 0.23, 0.06, and 0.05 at the first three lags. Thus, our model has the potential to generate predictability in implied returns.

Panel A of Table 3 shows that the AR(1) coefficient a_5 is estimated to be 0.14 with the AGARCH model and is statistically significant. Panel B reports a statistically significant estimate of 0.18 for simulated returns with the incomplete information stable model. All other panels show a very similar point estimate of a_5 . In every case, this coefficient is strongly significant. Thus, our model can reproduce observed predictability in implied returns.

We go on to examine the mean reversion behavior of returns if any (as in Fama and French, 1988a), their predictability from dividend yields (as in Fama and French, 1988b), and autocorrelations of the price–dividend ratios. For the sake of brevity we only report a brief summary of this analysis.

Regressions designed to test for mean reversion, as first done by Fama and French (1988a), yielded a U-shaped pattern for implied returns from our incomplete information stable model. This is consistent with the evidence reported in Fama and French (1988a) for US stock returns data. Plots of estimates of the variance ratio statistics, both for implied returns and those on quarterly value-weighted real returns on the NYSE, AMEX, and NASDAQ stocks, are similar. However, there is no evidence to suggest that our implied returns, especially at long horizons, are predictable from dividend yields, as has been found by Fama and French (1988b). Sample autocorrelations for our implied price–dividend ratios at the first five lags also match very closely those reported for US stocks data by Kurz et al. (2006).

5.5. Implications of stable distributions

In this section, we gather briefly all the implications of using stable distributions to model the unobservable dividends drift process in Eq. (6b). Because of the stability property of stable distributions (that sums of iid stable random variables are themselves stable), we are able to solve the asset pricing model with fat tails in a fairly simple way, with the price–dividend ratio given by the expression in Eq. (7). Any other fat-tailed distributional assumption would necessarily entail a more complicated approximate numerical solution to the asset pricing model.

Although the LR test in Section 4.3 fails to reject the Gaussian incomplete information model, as footnote 3 makes clear, such a model can be rejected statistically if one reverses the role of the stable and Gaussian distributions in Eqs. (6). For reasons elaborated in that footnote, we choose to retain the specification in Eqs. (6). As Fig. 4 makes clear, the stable model

delivers a time-varying filter variance, resulting in statistically significant time-varying volatility of implied returns, as in Table 3. On the other hand, the Gaussian setup delivers a constant filter variance, and hence cannot generate statistically significant time-varying volatility of implied returns. As Section 5.3 makes clear, the stable model results in implied returns that are not normally distributed. This conforms to the abundant empirical evidence on non-normality of returns data, as for instance in Bidarkota and McCulloch (2004).

6. Conclusions

We study the consumption based asset pricing model of Lucas (1978) in incomplete information setting with stable shocks driving the exogenous stochastic dividends growth rate process. The unobservability of the persistent component of dividends growth rates introduces a non-Gaussian filtering problem that agents solve using Bayesian updating schemes. This leads to a recursive estimate of the persistent component of the dividend growth rate, whose conditional variance always reacts to new data unlike the Kalman filter in the Gaussian setting. This time variation in the conditional variance of the agents' estimate of the persistent component leads to time variation in the volatility of implied returns.

We test this implication of our model using quarterly per capita real US consumption data. Our results indicate that the implied equilibrium returns display statistically significant volatility persistence of a magnitude comparable to that in the data. Our model also replicates leverage effect noted in the time-varying volatility of observed returns, although this result is not robust to changes in the distributional assumptions about implied returns. Neither incomplete information in a Gaussian setting, nor complete information in either Gaussian or stable settings, is able to generate these features in implied returns.

Acknowledgments

An earlier version of this paper authored by Prasad V. Bidarkota and J. Huston McCulloch was circulated as, "News or Noise? Signal Extraction can Generate Volatility Clusters from iid Shocks". The authors are grateful to participants at Florida International University, Melbourne, La Trobe, Ohio State, North American Summer Meetings of the Econometric Society, International Conference on Computing in Economics and Finance, FMA Annual Meeting, and to Pietro Veronesi, Vance Martin, Carl Chiarella (editor) and two anonymous referees for several helpful comments. They retain full responsibility for any remaining errors.

Appendix A. Evaluation of conditional expectations in the price–dividend ratio

In this appendix we derive conditions under which the conditional expectation terms that appear in Eq. (5) in the text are finite. We need to derive conditions under which

$$E_t \prod_{j=1}^i m_{t+j} < \infty,$$

where $m_{t+j} \equiv \theta \exp[(1 - \gamma)x_{t+j}]$.

Let $\omega \equiv 1 - \gamma$. Therefore, $m_{t+j} = \theta \exp[\omega x_{t+j}]$.

$$\prod_{j=1}^i m_{t+j} = \theta^i \exp\left(\omega \sum_{j=1}^i x_{t+j}\right). \quad (\text{A.1})$$

From dividend growth rate process in Eq. (6a),

$$\sum_{j=1}^i x_{t+j} = \sum_{j=1}^i \mu_{t+j} + \sum_{j=1}^i \varepsilon_{t+j}. \quad (\text{A.2})$$

From dividend growth rate process in Eq. (6b):

$$\sum_{j=1}^i \mu_{t+j} = i\bar{\mu} + (\mu_t - \bar{\mu}) \left[\frac{\rho(1 - \rho^i)}{1 - \rho} \right] + \frac{1}{1 - \rho} \left[(1 - \rho^i)\eta_{t+1} + (1 - \rho^{i-1})\eta_{t+2} + \dots + (1 - \rho)\eta_{t+i} \right]. \quad (\text{A.3})$$

Therefore,

$$\begin{aligned} \prod_{j=1}^i m_{t+j} &= \theta^i \exp\left(i\bar{\mu}\omega + \left\{ \frac{\omega\rho}{1 - \rho}(1 - \rho^i) \right\} (\mu_t - \bar{\mu}) + \left(\frac{\omega}{1 - \rho} \right) \right. \\ &\quad \left. \times \left\{ (1 - \rho^i)\eta_{t+1} + (1 - \rho^{i-1})\eta_{t+2} + \dots + (1 - \rho)\eta_{t+i} \right\} + \omega \sum_{j=1}^i \varepsilon_{t+j} \right). \end{aligned}$$

Define $b_i \equiv \omega(\rho/(1-\rho))(1-\rho^i)$. From the iid nature of ε_t and η_t , we can write:

$$E_t \prod_{j=1}^i m_{t+j} = \theta^i \exp[i\bar{\mu}\omega] E_t \exp[b_i(\mu_t - \bar{\mu})] E_t \left[\exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho^i)\eta_{t+1}\right] \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho^{i-1})\eta_{t+2}\right] \dots \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho)\eta_{t+i}\right] \right] E_t [\exp(\omega\varepsilon_{t+1}) \exp(\omega\varepsilon_{t+2}) \dots \exp(\omega\varepsilon_{t+i})]. \tag{A.4}$$

Since $\varepsilon_t \sim iid N(0, \sigma^2)$ in Eq. (6a),

$$E_t [\exp(\omega\varepsilon_{t+1}) \exp(\omega\varepsilon_{t+2}) \dots \exp(\omega\varepsilon_{t+i})] = E_t \{\exp(\omega\varepsilon_{t+1})\} E_t \{\exp(\omega\varepsilon_{t+2})\} \dots E_t \{\exp(\omega\varepsilon_{t+i})\}. \tag{A.5}$$

From the moment generating function of normal random variables, we have

$$E_t \{\exp(\omega\varepsilon_{t+1})\} = E_t \{\exp(\omega\varepsilon_{t+2})\} = \dots = E_t \{\exp(\omega\varepsilon_{t+i})\} = \exp\{\frac{1}{2}\omega^2\sigma^2\}. \tag{A.6}$$

Since $\eta_t \sim iid S(\alpha, \beta, c, 0)$ in Eq. (6b),

$$E_t \left\{ \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho^i)\eta_{t+1}\right] \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho^{i-1})\eta_{t+2}\right] \dots \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho)\eta_{t+i}\right] \right\} = E_t \left\{ \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho^i)\eta_{t+1}\right] \right\} E_t \left\{ \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho^{i-1})\eta_{t+2}\right] \right\} \dots E_t \left\{ \exp\left[\left(\frac{\omega}{1-\rho}\right)(1-\rho)\eta_{t+i}\right] \right\}. \tag{A.7}$$

Using Eq. (A.4) from Bidarkota and McCulloch (2003):

$$\frac{\omega}{1-\rho}(1-\rho^i)\eta_{t+1} \sim S\left(\alpha, \text{sign}\left\{\omega\left(\frac{1-\rho^i}{1-\rho}\right)\right\}\beta, \left|\omega\left(\frac{1-\rho^i}{1-\rho}\right)\right|c, 0\right).$$

Since Eq. (6b) also specifies that $|\rho| < 1$, we have $(1-\rho^i)/(1-\rho) > 0$. Therefore,

$$\text{sign}\left\{\omega\left(\frac{1-\rho^i}{1-\rho}\right)\right\} = \text{sign}(\omega).$$

Hence,

$$\frac{\omega}{1-\rho}(1-\rho^i)\eta_{t+1} \sim S\left(\alpha, \text{sign}(\omega)\beta, \left|\omega\left(\frac{1-\rho^i}{1-\rho}\right)\right|c, 0\right) \tag{A.8}$$

and so forth for all the other η 's in Eq. (A.7).

Now,

$$E_t \prod_{j=1}^i m_{t+j} < \infty$$

if the right hand-side of Eq. (A.4) is finite. This requires that each of the three conditional expectation terms on the right hand-side of Eq. (A.4) be finite. From Eq. (A.6), the third conditional expectation term on the right hand-side of Eq. (A.4) is finite.

The second conditional expectation term on the right hand-side of Eq. (A.4) is finite if each of the conditional expectation terms on the right hand-side of Eq. (A.7) is finite. Using Eq. (A.8) above and (A.8) of Bidarkota and McCulloch (2003), this happens when $\beta \text{sign}(\omega) = -1$. Or, substituting for ω , this happens when $\beta \text{sign}(1-\gamma) = -1$.

From Eq. (6b), we can solve for $\mu_t - \bar{\mu}_0$ as

$$\mu_t - \bar{\mu} = \sum_{i=0}^{\infty} \rho^i \eta_{t-i}. \tag{A.9}$$

Eq. (6b) also specifies that $\eta_t \sim iid S(\alpha, \beta, c, 0)$. Then, using Eq. (A.4) from Bidarkota and McCulloch (2003) and the fact that $0 \leq \rho$, we have $\rho^i \eta_{t-i} \sim iid S(\alpha, \beta, \rho^i c, 0)$.

Using Eqs. (A.5), (A.6), and (A.7) from Bidarkota and McCulloch (2003), we get

$$\sum_{i=1}^{\infty} \rho^i \eta_{t-i} \sim S\left(\alpha, \beta, \left|\sum_{i=0}^{\infty} (\rho^i)^z\right|^{1/z}, c, 0\right).$$

With $|\rho| < 1$ already assumed in Eq. (6b),

$$\sum_{i=0}^{\infty} (\rho^i)^z = \frac{1}{1-\rho^z}.$$

Therefore, from Eq. (A.9),

$$\mu_t - \bar{\mu} = \sum_{i=0}^{\infty} \rho^i \eta_{t-i} \sim S\left(\alpha, \beta, \frac{c}{[1 - \rho^\alpha]^{1/\alpha}}, 0\right). \tag{A.10}$$

From the definition of $b_i \equiv \omega(\rho/1 - \rho)(1 - \rho^i)$, $\text{sign}(b_i) = \text{sign}(\omega)$. Therefore,

$$b_i(\mu_t - \bar{\mu}) \sim S\left(\alpha, \text{sign}(\omega)\beta, \frac{|b_i|}{(1 - \rho^\alpha)^{1/\alpha}} c, 0\right). \tag{A.11}$$

The unconditional expectation $E[\exp\{b_i(\mu_t - \bar{\mu})\}]$ is finite if $\beta \text{sign}(\omega) = \beta \text{sign}(1 - \gamma) = -1$ from Eq. (A.8) of Bidarkota and McCulloch (2003). It follows from the law of iterated expectations that the first conditional expectation term on the right hand-side of Eq. (A.4) is finite if $\beta \text{sign}(1 - \gamma) = -1$.

Therefore,

$$E_t \prod_{j=1}^i m_{t+j} < \infty,$$

when $\beta \text{sign}(1 - \gamma) = -1$.

Appendix B. Derivation of the tractable expression for the price–dividend ratio

In this appendix we derive the expression for the price dividend ratio v_t given in Eq. (7). Eq. (5) gives

$$v_t = \sum_{i=1}^{\infty} \left(E_t \prod_{j=1}^i m_{t+j} \right). \tag{B.1}$$

From Appendix A, we know that

$$E_t \prod_{j=1}^i m_{t+j} < \infty,$$

when $\beta \text{sign}(1 - \gamma) = -1$.

Under $\beta = -1$ and $\gamma < 1$ and using $|\rho| < 1$, we get from Eqs. (A.9), (A.10) and (A.8) of Bidarkota and McCulloch (2003):

$$E_t \left\{ \exp \left[\left(\frac{\omega}{1 - \rho} \right) (1 - \rho^i) \eta_{t+1} \right] \right\} = \exp \left[- \left\{ \omega \left(\frac{1 - \rho^i}{1 - \rho} \right) \right\}^\alpha c^\alpha \sec \left(\frac{\pi\alpha}{2} \right) \right], \tag{B.2}$$

$$E_t \left\{ \exp \left[\left(\frac{\omega}{1 - \rho} \right) (1 - \rho^{i-1}) \eta_{t+2} \right] \right\} = \exp \left[- \left\{ \omega \left(\frac{1 - \rho^{i-1}}{1 - \rho} \right) \right\}^\alpha c^\alpha \sec \left(\frac{\pi\alpha}{2} \right) \right] \tag{B.3}$$

and so forth for all the other η 's in Eq. (A.8):

$$E_t \left\{ \exp \left[\left(\frac{\omega}{1 - \rho} \right) (1 - \rho) \eta_{t+i} \right] \right\} = \exp \left[- \left\{ \omega \left(\frac{1 - \rho}{1 - \rho} \right) \right\}^\alpha c^\alpha \sec \left(\frac{\pi\alpha}{2} \right) \right]. \tag{B.4}$$

Substituting (B.2), (B.3), and (B.4) into (A.8), we get

$$\begin{aligned} E_t \left\{ \exp \left[\left(\frac{\omega}{1 - \rho} \right) (1 - \rho^i) \eta_{t+1} \right] \exp \left[\left(\frac{\omega}{1 - \rho} \right) (1 - \rho^{i-1}) \eta_{t+2} \right] \dots \exp \left[\left(\frac{\omega}{1 - \rho} \right) (1 - \rho) \eta_{t+i} \right] \right\} \\ = \exp \left\{ - \left[\left\{ \frac{\omega}{1 - \rho} \right\}^\alpha c^\alpha \sec \left(\frac{\pi\alpha}{2} \right) \right] \sum_{j=1}^i (1 - \rho^j)^\alpha \right\}. \end{aligned} \tag{B.5}$$

Substituting (B.5) and (A.7) into (A.5) and collecting terms results in:

$$\begin{aligned} E_t \prod_{j=1}^i m_{t+j} = \theta^i [E_t \exp\{b_i(\mu_t - \bar{\mu})\}] \exp \left[i\bar{\mu}(1 - \gamma) + i(1 - \gamma)^2 \frac{\sigma^2}{2} \right. \\ \left. + \left\{ - \left[\left\{ \frac{1 - \gamma}{1 - \rho} \right\}^\alpha c^\alpha \sec \left(\frac{\pi\alpha}{2} \right) \right] \sum_{j=1}^i (1 - \rho^j)^\alpha \right\} \right] \end{aligned} \tag{B.6}$$

recognizing from Appendix A that $\omega = 1 - \gamma$.

Finally, substituting (B.6) into (B.1) gives

$$v_t = \sum_{i=1}^{\infty} \theta^i [E_t \exp(b_i(\mu_t - \bar{\mu}))] \exp \left[i\bar{\mu}(1 - \gamma) + i(1 - \gamma)^2 \frac{\sigma^2}{2} \right. \\ \left. + \left\{ - \left[\left\{ \frac{1 - \gamma}{1 - \rho} \right\}^\alpha c^\alpha \sec \left(\frac{\pi\alpha}{2} \right) \right] \sum_{j=1}^i (1 - \rho^j)^\alpha \right\} \right], \quad (\text{B.7})$$

where, from Appendix A, we have $b_i \equiv (1 - \gamma)(\rho/1 - \rho)(1 - \rho^i)$.

References

- Andrews, D.W.K., 2001. Testing when a parameter is on the boundary of the maintained hypothesis. *Econometrica* 69 (3), 683–734.
- Arrow, K.J., 1971. *Essays in the Theory of Risk Bearing*. North-Holland, Amsterdam.
- Bansal, R., Yaron, A., 2004. Risks for the long run: a potential resolution of asset pricing puzzles. *Journal of Finance* 59 (4), 1481–1509.
- Bidarkota, P.V., 2001. Alternative regime switching models for forecasting inflation. *Journal of Forecasting* 20, 21–35.
- Bidarkota, P.V., 2003. Do fluctuations in U.S. inflation rates reflect infrequent large shocks or frequent small shocks? *Review of Economics and Statistics* 85 (3), 765–771.
- Bidarkota, P.V., Dupoyet, B.V., 2007. The impact of fat tails on equilibrium rates of return and term premia. *Journal of Economic Dynamics and Control* 31, 887–905.
- Bidarkota, P.V., McCulloch, J.H., 2004. Testing for persistence in stock returns with GARCH-stable shocks. *Quantitative Finance* 4 (3), 256–265.
- Bidarkota, P.V., McCulloch, J.H., 2003. Consumption asset pricing with stable shocks: exploring a solution and its implications for mean equity returns. *Journal of Economic Dynamics and Control* 27 (3), 399–421.
- Bidarkota, P.V., McCulloch, J.H., 1998. Optimal univariate inflation forecasting with symmetric stable shocks. *Journal of Applied Econometrics* 13, 659–670.
- Brandt, M.W., Zeng, Q., Zhang, L., 2004. Equilibrium stock return dynamics under alternative rules of learning about hidden states. *Journal of Economic Dynamics and Control* 28, 1925–1954.
- Brennan, M.J., Xia, Y., 2001. Stock price volatility and equity premium. *Journal of Monetary Economics* 47, 249–283.
- Burnside, C., 1998. Solving asset pricing models with Gaussian shocks. *Journal of Economic Dynamics and Control* 22, 329–340.
- Carr, P., Wu, L., 2003. The finite moment log-stable process and option pricing. *Journal of Finance* LVIII (2), 753–777.
- David, A., 1997. Fluctuating confidence in stock markets: implications for returns and volatility. *Journal of Financial and Quantitative Analysis* 32 (4), 427–462.
- Detemple, J.B., 1991. Further results on asset pricing with incomplete information. *Journal of Economic Dynamics and Control* 15, 425–453.
- Detemple, J.B., 1986. Asset pricing in a production economy with incomplete information. *Journal of Finance* 41 (3), 383–391.
- Dothan, M.U., Feldman, D., 1986. Equilibrium interest rates and multiperiod bonds in a partially observable economy. *Journal of Finance* 41 (2), 369–382.
- Durbin, J., Koopman, S.J., 2000. Time series analysis of non-Gaussian observations based on state space models from both classical and Bayesian perspectives. *Journal of the Royal Statistical Society, Series B* 62 (Part 1), 3–56.
- Epstein, L.G., Zin, S., 1989. Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework. *Econometrica* 57, 937–969.
- Epstein, L.G., Zin, S., 1991. Substitution, risk aversion and the temporal behavior of consumption and asset returns: an empirical analysis. *Journal of Political Economy* 99, 263–268.
- Fama, E.F., French, K.R., 1988a. Permanent and temporary components of stock prices. *Journal of Political Economy* 96, 246–273.
- Fama, E.F., French, K.R., 1988b. Dividend yields and expected stock returns. *Journal of Financial Economics* 22, 3–25.
- Gennotte, G., 1986. Optimal portfolio choice under incomplete information. *Journal of Finance* 41, 733–746.
- Geweke, J., 2001. A note on some limitations of CRRA utility. *Economics Letters* 71, 341–345.
- Hansen, L.P., Singleton, K.J., 1982. Generalized instrumental variables estimation of nonlinear rational expectations models. *Econometrica* 50, 1269–1286.
- Harvey, A.C., 1992. *Forecasting, Structural Time Series Models and the Kalman Filter*. Cambridge University Press, Cambridge, UK.
- Hill, B.M., 1975. A simple general approach to inference about the tail of a distribution. *Annals of Statistics* 3, 1163–1174.
- Kearns, P., Pagan, A., 1997. Estimating the density tail index for financial time series. *The Review of Economics and Statistics* 79 (2), 171–175.
- Kitagawa, G., 1987. Non-Gaussian state space modeling of nonstationary time series. *Journal of the American Statistical Association* 82 (400), 1032–1063.
- Kurz, M., Jin, H., Motolese, M., 2006. Determinants of stock market volatility and risk premia. *Annals of Finance* 1, 109–147.
- Lucas Jr., R.E., 1978. Asset prices in an exchange economy. *Econometrica* 46, 1429–1445.
- Malik, F., Ewing, B.T., Payne, J.E., 2005. Measuring volatility persistence in the presence of sudden changes in the variance of Canadian stock returns. *Canadian Journal of Economics* 38 (3), 1037–1056.
- Mandelbrot, B., 1963. The variation of certain speculative prices. *Journal of Business* 36, 394–419.
- McCulloch, J.H., 1997. Measuring tail thickness in order to estimate the stable index α : a critique. *Journal of Business and Economic Statistics* 15 (1), 74–81.
- McCulloch, J.H., 1996. Financial applications of stable distributions. In: Maddala, G.S., Rao, C.R. (Eds.), *Handbook of Statistics*, vol. 14. Elsevier, Amsterdam, pp. 393–425.
- Mehra, R., Prescott, E.C., 1985. The equity premium: a puzzle. *Journal of Monetary Economics* 15, 145–161.
- Mittnik, S., Doganoglu, T., Chenyao, D., 1999. Computing the probability density function of the stable paretian distribution. *Mathematical and Computer Modelling* 29, 235–240.
- Sorenson, H.W., Alspach, D.L., 1971. Recursive Bayesian estimation using Gaussian sums. *Automatica* 7, 465–479.
- Tsiakos, I., 2006. Periodic stochastic volatility and fat tails. *Journal of Financial Econometrics* 4 (1), 90–135.
- Veronesi, P., 2004. Belief dependent utilities, aversion to state uncertainty, and asset prices. Unpublished manuscript, GSB University of Chicago.
- Veronesi, P., 1999. Stock market overreaction to bad news in good times: a rational expectations equilibrium model. *Review of Financial Studies* 12 (5), 975–1007.
- Wu, L., 2006. Dampened power law: reconciling the tail behavior of financial security returns. *Journal of Business* 79 (3), 1445–1474.