Physica A 389 (2010) 107–116

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Gauge invariant lattice quantum field theory: Implications for statistical properties of high frequency financial markets

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ARTICLE INFO

Article history:
Received 24 April 2009
Received in revised form 12 August 2009
Available online 6 September 2009

PACS:
89.65.Gh
11.15.Ha

Keywords:
Econophysics
Financial markets
Statistical field theory
Gauge invariance

ABSTRACT

We report on initial studies of a quantum field theory defined on a lattice with multi-ladder geometry and the dilation group as a local gauge symmetry. The model is relevant in the cross-disciplinary area of econophysics. A corresponding proposal by Ilinski aimed at gauge modeling in non-equilibrium pricing is implemented in a numerical simulation. We arrive at a probability distribution of relative gains which matches the high frequency historical data of the NASDAQ stock exchange index.

Published by Elsevier B.V.

1. Introduction

The concept of a gauge theory revolves around the notion that the laws of physics must be independent of arbitrary choices made by an observer, such as the scales (units) of length and time, or the choice of some coordinates in general.

Gauge invariance came into prominence in 1929 through the work of Weyl [1] in a quest to understand gravitation. Since then, it has evolved beyond its original context to also become a cornerstone of modern quantum field theory. The gauge principle is realized in terms of a local symmetry group G which reflects some aspect of the physics. Famously, quantum electrodynamics (QED) is locally gauge invariant with respect to $G = U(1)$, meaning that the choice of the complex phase for matter fields is arbitrary. Indeed, our current understanding of particle physics as a unified theory of electroweak and strong interactions, the so-called standard model, is built on a gauge principle combined with a specific pattern for (spontaneous) symmetry breaking.

Methods of theoretical physics have found their way into financial mathematics and related fields of economics [2]. The term "econophysics" has been coined to describe this area of research [3]. A typical application is concerned with stochastic properties of historical (market) data. This explains the connection to quantum physics, which is based on a stochastic interpretation of physical phenomena. The marriage of financial mathematics with quantum physics has been called quantum finance [4]. Quantization is best achieved using path integrals [5] because this guarantees viability even in nonperturbative systems, like off-equilibrium markets. Moreover, numerical simulation on some lattice geometry is readily implemented.

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0378-4371/$ – see front matter. Published by Elsevier B.V.
doi:10.1016/j.physa.2009.09.002

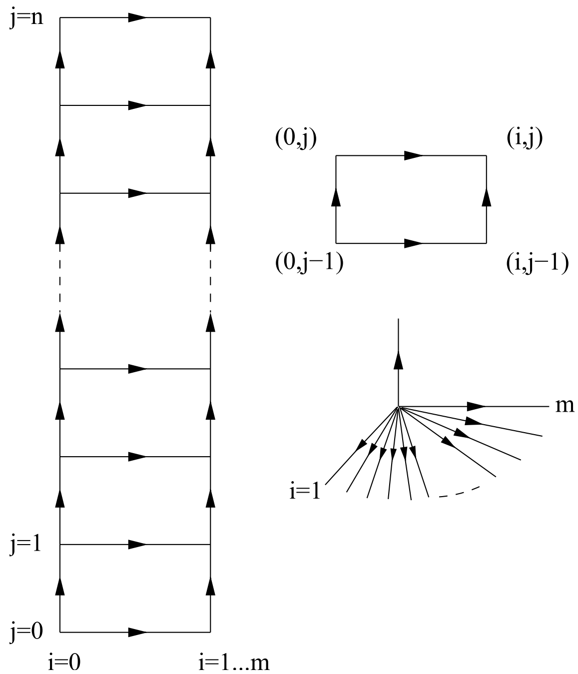

Fig. 1. Illustration of the geometry of the lattice model and the label scheme for the sites.

The idea of combining the gauge principle with quantum field theory in the context of econophysics has been advanced by Ilinski [6]. Its rationale is that traders in financial markets, for example, interact in similar ways with the trading environment, irrespective of the units used for the traded commodity. For example, market observables like the distribution of relative gains to an investor should be similar regardless of whether the currency unit used in the exchange is USD, Euro, Yen, or some other unit, provided that an appropriate scale factor has been applied. Empirical evidence related to this idea comes from the observation of scaling of the market data with respect to different time horizons [7], yet another scale.

A somewhat independent path of using the concept of a gauge in stochastic investment models in mathematical finance has been pursued by Smith and Speed [8] and Farinelli [9]. The concepts of differential geometry and stochastic mathematics are combined in a rather formal way to yield interesting theoretical results. Comparing our and Ilinski's [6] approaches requires a translation of terminology: The terms "deflator and term structure" refer to "gauge fields", though in principle the concepts are similar. We prefer Ilinski's approach over Farinelli's, in part because it more directly lends itself to practical application.

The present paper is concerned with an implementation of those ideas in the framework of a numerical simulation of a simple gauge invariant lattice quantum field theory. Subsequently, we will describe the lattice model, its interpretation in the context of financial markets, and the numerical implementation, and then proceed to matching the results to historical data of the NASDAQ stock index at one-minute intervals.

2. The lattice model

Inspired by Ref. [6] we work with a lattice geometry as illustrated in Fig. 1. In a three-dimensional rendering, the vertical direction represents time with slices $j = 0, 1, \ldots, n$ separated by some arbitrary unit. At each time slice, we imagine a horizontal plane in which there are space locations. A second index $i = 0$ labels the location of the origin and $i = 1, \ldots, m$ indicates (different) locations one step away from it. It is useful to divide the lattice sites into two sets

$$\mathcal{X}_0 = \{(0, j) | j = 0, \ldots, n\} \tag{1}$$

$$\mathcal{X}_1 = \{(i, j) | i = 1, \ldots, m, j = 0, \ldots, n\}. \tag{2}$$

The sites are connected through three sets of links. In the space direction (horizontal) we have

$$\mathcal{L}^s = \{(0, j) \to (i, j) | i = 1 \ldots m, j = 0, \ldots, n\}, \tag{3}$$

and in the time direction (vertical), we distinguish two sets of links

$$\mathcal{L}_0^t = \{(0, j - 1) \to (0, j) | j = 1, \ldots, n\} \tag{4}$$

$$\mathcal{L}_1^t = \{(i, j - 1) \to (i, j) | i = 1, \ldots, m, j = 1, \ldots, n\}. \tag{5}$$

The lattice geometry may be visualized by imagining m copies of a ladder, with $n+1$ rungs each, that have their left rails in common, which then consists of $\mathcal{X}_0 \cup \mathcal{L}_0^t$. Note that sites on the right rails of different ladders are not connected by single links. There are $(n+1)m$ horizontal (space) links and $n(m+1)$ vertical (time) links connecting a total of $(n+1)(m+1)$ sites.

It will be convenient to visualize the geometry defined above as being embedded in an infinite lattice of dimension $d = 1 + m$. There, we label the directions by $\mu = 0, 1, \ldots, m$ and define unit vectors e_μ in terms of their components k by

$$(e_\mu)_k = \delta_{\mu k} \quad \text{where } k = 0, 1, \ldots, m. \tag{6}$$

Then, the sites (i, j) of our ladder geometry may be mapped into the sites $x = \sum_\mu x_\mu e_\mu$, where $x_\mu \in \mathbb{Z}$, as

$$\mathcal{X}_0 \ni (0, j) \to je_0 \tag{7}$$

$$\mathcal{X}_1 \ni (i, j) \to je_0 + e_i. \tag{8}$$

We populate the lattice with a matter field Φ, its conjugate $\bar{\Phi}$, and a gauge field Θ. The matter fields live on the lattice sites $\mathcal{X} = \mathcal{X}_0 \cup \mathcal{X}_1$ and assume values in the set of real positive numbers $\mathbb{R}^+$. The conjugate field is not independent of Φ. It is introduced for convenience of notation; we have

$$\Phi(x), \bar{\Phi}(x) \in \mathbb{R}^+ \quad \text{with } \bar{\Phi}(x) = 1/\Phi(x). \tag{9}$$

The gauge field is defined on the oriented links $\mathcal{L} = \mathcal{L}^s \cup \mathcal{L}_0^t \cup \mathcal{L}_1^t$. We write $\Theta_\mu(x)$ for a link variable, meaning that the link originates at site x and terminates at site $x + e_\mu$. The range is

$$\Theta_\mu(x) \in \mathbb{R}^+. \tag{10}$$

The goal is to construct a locally gauge invariant theory based on the dilation group; i.e. the set of positive real numbers with ordinary multiplication of the group elements

$$G = \{g | g \in \mathbb{R}^+\} \quad \text{with } g_1 \circ g_2 = g_1 g_2. \tag{11}$$

Under a local gauge transformation $g(x) \in G$ the fields behave as

$$\Phi(x) \to g(x)\Phi(x) \tag{12}$$

$$\bar{\Phi}(x) \to \bar{\Phi}(x)g^{-1}(x) \tag{13}$$

$$\Theta_\mu(x) \to g(x)\Theta_\mu(x)g^{-1}(x + e_\mu). \tag{14}$$

The quantum field theory of our system will be governed by an action $S[\Theta, \Phi, \bar{\Phi}]$, which is a functional constructed from gauge invariant combinations of the fields. For gauge fields only, the most local (smallest) gauge invariant construct is the so-called elementary plaquette

$$P_{\mu\nu}(x) = \Theta_\mu(x)\Theta_\nu(x + e_\mu)\Theta_\mu^{-1}(x + e_\nu)\Theta_\nu^{-1}(x), \tag{15}$$

where $\nu = i > 0$ and $\mu = 0$ on our lattice geometry, as illustrated in Fig. 1. As a closed loop, it is invariant under a local gauge transformation (14)

$$P_{\mu\nu}(x) \to P_{\mu\nu}(x). \tag{16}$$

Other building blocks involve covariant derivatives of the matter fields. Possible choices are

$$D_\mu^+\Phi(x) = \Theta_\mu(x)\Phi(x + e_\mu) - \Phi(x) \tag{17}$$

$$D_\mu^-\Phi(x) = \Theta_\mu^{-1}(x - e_\mu)\Phi(x - e_\mu) - \Phi(x) \tag{18}$$

$$\bar{D}_\mu^+\bar{\Phi}(x) = \bar{\Phi}(x + e_\mu)\Theta_\mu^{-1}(x) - \bar{\Phi}(x) \tag{19}$$

$$\bar{D}_\mu^-\bar{\Phi}(x) = \bar{\Phi}(x - e_\mu)\Theta_\mu(x - e_\mu) - \bar{\Phi}(x). \tag{20}$$

It is straightforward to check that these transform under local gauge transformations (12)–(14) as

$$D_\mu^\pm\Phi(x) \to g(x)D_\mu^\pm \tag{21}$$

$$\bar{D}_\mu^\pm\bar{\Phi}(x) \to \bar{D}_\mu^\pm\Phi(x)g^{-1}(x). \tag{22}$$

Thus, the simplest (smallest) locally gauge invariant constructs are $\bar{\Phi}(x)D_\mu^\pm\Phi(x)$ and $\bar{D}_\mu^\pm\bar{\Phi}(x)\Phi(x)$. All of these contain terms $\propto \bar{\Phi}(x)\Phi(x)$, which are gauge invariant. However, because of (9) these are constant and do not affect the dynamics of the fields.[1] Thus, dropping those terms, a locally gauge invariant action for the quantum field can be written as

$$S[\Theta, \Phi, \bar{\Phi}] = S_0[\Theta] + S_1[\Theta, \Phi, \bar{\Phi}] \tag{23}$$

[1] In the case of a complex field $\Phi \in \mathbb{C}$ this would be known as a mass term.

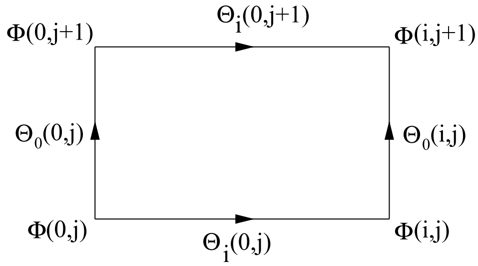

Fig. 2. Arbitrage through an elementary plaquette.

with

$$S_0[\Theta] = \frac{1}{2}\sum_x \sum_{\mu<\nu} [P_{\mu\nu}(x) + P_{\nu\mu}(x) - 2] \tag{24}$$

$$\begin{aligned} S_1[\Theta, \Phi, \bar{\Phi}] \ = \ & \sum_x \sum_\mu [d_\mu^+ \bar{\Phi}(x)\Theta_\mu(x)\Phi(x+e_\mu) + d_\mu^- \bar{\Phi}(x)\Theta_\mu^{-1}(x-e_\mu)\Phi(x-e_\mu) + \bar{d}_\mu^+ \bar{\Phi}(x+e_\mu)\Theta_\mu^{-1}(x)\Phi(x) \\ & + \bar{d}_\mu^- \bar{\Phi}(x-e_\mu)\Theta_\mu(x-e_\mu)\Phi(x)]. \end{aligned} \tag{25}$$

The coupling constants $d_\mu^\pm, \bar{d}_\mu^\pm \in \mathbb{R}$ determine the interactions of matter and gauge fields. In (24) and (25) it is understood that the summations comprise only sites and links contained in the lattice geometry described above; see Fig. 1.

We use the well-established formalism of lattice quantum field theory in Euclidean space-time to quantize the system; see for example Ref. [10]. The fields Θ, Φ are taken to be random variables with a Boltzmann-like probability distribution $\propto \exp(-\beta S)$ where β is a parameter. The corresponding partition function then is defined as the functional integral

$$Z(\beta) = \int [D\Theta][D\Phi] \mathrm{e}^{-\beta S[\Theta,\Phi,\bar{\Phi}]}. \tag{26}$$

We defer a discussion of technical features to a later section, and continue with placing the model in a financial market context.

3. Giving meaning to the fields

A gauge model based on the dilation group as a tool for pricing financial instruments was discussed extensively by Ilinski [6]. The author draws an analogy to quantum electrodynamics, although a generic one, and even briefly touches on a lattice version of the model. The geometry is ladder-like too; however, it differs slightly from the geometry used here in the way the links connect to the lattice sites. Our version is well suited to numerical simulation.

We adopt the interpretation of the fields from Ref. [6]. In the labeling scheme of (1)–(5), and Fig. 1, space locations i are associated with accounts held by an investor. The location $i = 0$ is special; it denotes a cash account maintained to manage transactions. The field component $\Phi(0, j)$ is the balance of the cash account at time j. The unit of $\Phi(0, j)$ is that of some arbitrary currency; USD for example. Locations $i > 0$ indicate holdings in investment instruments. The meaning of $\Phi(i, j)$ is the account value of instrument i at time j. Again, the units are arbitrary. For example, one may think of the market value of all shares held in the account. It is useful to refer to Fig. 2 for this discussion.

For $i > 0$, the sites $(0, j)$ and (i, j) are joined by horizontal links $\in \mathcal{L}^s$. The corresponding gauge field components $\Theta_i(0, j)$ are interpreted as conversion factors between the units used at locations $i = 0$ and those at $i > 0$. In the language of differential geometry, the field is known as a connection. We say that $\Theta_i(0, j)$ performs a parallel transport of the matter field at (i, j) so it can be compared to the matter field at $(0, j)$ in like units. Thus, $\Theta_i(0, j)\Phi(i, j)$ and $\Phi(0, j)$ have the same units.

Gauge fields defined on vertical links $\in \mathcal{L}_0^t$ are interpreted as interest rate factors. For example, we may imagine that $i = 0$ describes a money market account. A cash balance $\Phi(0, j)$ at time j will assume a value $\Phi(0, j + 1)$ at time $j + 1$. It can be compared to $\Phi(0, j)$ only after parallel transport, namely $\Theta_0(0, j)\Phi(0, j + 1)$. In finance the latter quantity is known as the discounted cash value. The link variable involves the interest rate.

Likewise, the gauge fields defined on vertical links $\in \mathcal{L}_1^t$ lend themselves, mutatis mutandis, to the same interpretation. For $i > 0$ the value of instrument i will change from $\Phi(i, j)$ to $\Phi(i, j + 1)$ driven by the market environment. A comparison to $\Phi(i, j)$ can be made with $\Theta_0(i, j)\Phi(i, j + 1)$, i.e. after parallel transport to the same time slice.

An investor holding a cash balance of $\Phi(0, j)$ at time j may wish to make a profit by utilizing an investment instrument. The investor may buy shares in the instrument $i > 0$, wait for one time step and then sell it, depositing the cash amount back into $i = 0$. The properly discounted cash value at time j then is

$$V_1 = \Theta_i(0, j)\Theta_0(i, j)\Theta_i^{-1}(0, j + 1)\Phi(0, j + 1). \tag{27}$$

If the cash balance had been kept in account $i = 0$, its discounted value at time j would have been

$$V_0 = \Theta_0(0, j)\Phi(0, j+1). \tag{28}$$

The relative gain (loss) of those two moves then is

$$\begin{aligned} V_1/V_0 &= \bar{\Phi}(0, j+1)\Theta_0^{-1}(0, j)\Theta_i(0, j)\Theta_0(i, j)\Theta_i^{-1}(0, j+1)\Phi(0, j+1) \\ &= \Theta_i(0, j)\Theta_0(i, j)\Theta_i^{-1}(0, j+1)\Theta_0^{-1}(0, j) \\ &= P_{\mu\nu}(x), \end{aligned} \tag{29}$$

where in the last line we have indicated that the relative gain is equal to an elementary plaquette (15) with $\mu = i$ and $\nu = 0$. Such a profit opportunity is known as arbitrage. In the real world, those will quickly vanish because the market participants will adjust prices. The gauge field model is designed to describe this situation [6], because the gauge field action $S_0[\Theta]$ is built from elementary plaquettes (24). The classical minimum is clearly realized at $P_{\mu\nu}(x) = 1$, for all μ, ν, x, which means that there is no arbitrage opportunity at the classical level. However, for short periods of time, arbitrage opportunities do exist, and this is reflected in the quantum nature of the model. Quantum fluctuations about the classical minimum do occur with a probability given by (26). The sizes of these fluctuations are determined by the parameter β.

4. Numerical simulation

The expected value $\langle \mathcal{O} \rangle$ of an operator $\mathcal{O}[\Theta, \Phi]$, depending on the fields, is given by the functional integral

$$\langle \mathcal{O} \rangle = Z(\beta)^{-1} \int [D\Theta][D\Phi] \mathrm{e}^{-\beta S[\Theta, \Phi, \bar{\Phi}]} \mathcal{O}[\Theta, \Phi]. \tag{30}$$

We shall later consider operators $\mathcal{O}$ which represent the distribution of gains for financial instruments.

A few comments could be helpful to clarify the connection between the choice of gauge and the integration variables in (30). The path integral involves only gauge invariant quantities, specifically the Haar measure of the gauge group G, and the action S. We shall consider only gauge invariant operators $\mathcal{O}$. Henceforth, the expected value $\langle \mathcal{O} \rangle$ is independent of the choice of gauge. Since the fields live on a discrete finite lattice the path integral is also finite. We shall use a numerical estimator to obtain (30). Then the role of the integration variables is assumed by performing an average over a (truncated) Gibbs ensemble of field configurations. One could enforce a particular gauge condition at this point, but this is unnecessary. The estimator of the expected value $\langle \mathcal{O} \rangle$ is the same in any gauge.

The numerical simulation consists of computing a Gibbs ensemble of field configurations drawn from the probability distribution $\propto \mathrm{e}^{-\beta S[\Theta, \Phi, \bar{\Phi}]}$. More precisely, the ensemble is truncated to a finite set of configurations, rendering it an approximation. We take advantage of the tool set developed in the context of lattice quantum chromodynamics (LQCD), which describes the physics of strong interactions (between quarks, gluons, hadrons, etc.); see for example Ref. [10]. Compared to the LQCD case, our task is greatly simplified because the matter fields only take values in the set of real numbers, as opposed to (anticommuting) Grassmann numbers, which cause serious numerical complications.

In practice, we start with an arbitrary (e.g. random) field configuration and then generate a Markov chain of equilibrium configurations based on the Metropolis algorithm [11]. This yields a canonical ensemble of field configurations $[\Theta, \Phi]_k$, which we truncate, $k = 1, \ldots, K$, and use to estimate expectation values, such as (30), as a configuration average.

Our method of choice for numerically generating field configurations is the heat bath algorithm [10,12]. This is a standard method in numerical simulation, so we only give a brief sketch pertaining to the current implementation. The algorithm consists of locally updating the fields.

For a given gauge field link variable $\Theta_\mu(x)$, the action (23) can be written as a sum $S = \Delta S_\Theta + \bar{S}_\Theta$, where ΔS_Θ contains only terms that depend on $\Theta_\mu(x)$, and $\bar{S}_\Theta$ is independent of $\Theta_\mu(x)$. It is easy to see that

$$\Delta S_\Theta = \bar{L}_\Theta \Theta_\mu(x) + \Theta_\mu^{-1}(x) L_\Theta, \tag{31}$$

where $\bar{L}_\Theta$ and L_Θ are independent of $\Theta_\mu(x)$. These factors are made up from the gauge link staples connecting to $\Theta_\mu(x)$, and the matter fields at both ends of $\Theta_\mu(x)$. The Haar measure for the dilation group (11) is $d \ln g$, $g \in G$. Thus, we conveniently write

$$\Theta_\mu(x) = \exp(\theta_\mu(x)), \quad \theta_\mu(x) \in \mathbb{R}, \tag{32}$$

and observe that the probability density function for $\theta_\mu(x)$ is

$$p_\Theta(\theta_\mu(x)) \propto \exp(-\beta(\bar{L}_\Theta \exp(\theta_\mu(x)) + \exp(-\theta_\mu(x)) L_\Theta)). \tag{33}$$

Similarly, to update a given matter field component $\Phi(x)$, the action (23) can be written as a sum $S = \Delta S_\Phi + \bar{S}_\Phi$, where ΔS_Φ contains only terms that depend on $\Phi(x)$, and $\bar{S}_\Phi$ is independent of $\Phi(x)$. Then

$$\Delta S_\Phi = \bar{L}_\Phi \Phi(x) + \bar{\Phi}(x) L_\Phi, \tag{34}$$

where $\bar{L}_\Phi$ and L_Φ are independent of $\Phi(x)$. These factors are made up from gauge links and matter fields that are next neighbors to $\Phi(x)$. Writing

$$\Phi(x) = \exp(\phi(x)), \qquad \bar{\Phi}(x) = \exp(-\phi(x)), \quad \phi(x) \in \mathbb{R}, \tag{35}$$

we arrive at the probability density function for $\phi(x)$;

$$p_\Phi(\phi(x)) \propto \exp(-\beta(\bar{L}_\Phi \exp(\phi(x)) + \exp(-\phi(x))L_\Phi)). \tag{36}$$

The coefficients $\bar{L}_\Theta, L_\Theta$ and $\bar{L}_\Phi, L_\Phi$ depend on the local environment of the field components considered for updating. However, their detailed mathematical structure is such that the requirement

$$\bar{d}^+_\mu + d^-_\mu > 0 \quad \text{and} \quad \bar{d}^-_\mu + d^+_\mu > 0 \tag{37}$$

for the coupling constants (see (25)) is a sufficient condition for $\bar{L}_\Theta, L_\Theta > 0$ and $\bar{L}_\Phi, L_\Phi > 0$. Hence, the local probability density functions (33) and (36) are normalizable and bounded. In framework of the heat bath algorithm, we may then use the accept–reject method to generate sampling values from p_Θ and p_Φ.

5. Observables and results

In keeping with the interpretation of the model put forward in Section 3 we consider the operators

$$W_\ell(i,j) = \bar{\Phi}(i,j-\ell) \prod_{k=1}^{\ell} \Theta_0(i,j-k)\, \Phi(i,j). \tag{38}$$

For an account i they describe the ratio of its values at times j and $j-\ell$. The product of link variables along the time direction in (38) provides parallel transport, so the comparison is made in like units. The operators $W_\ell(i,j)$ are gauge invariant by construction. It is convenient to define

$$\Lambda_\ell(i,j) = \log W_\ell(i,j), \tag{39}$$

which is interpreted as (the logarithm of) the relative profit made via the instrument in account i, at time j, after having held it for a time interval of length ℓ. In a quantum setting, the expectation values $\langle \Lambda_\ell(i,j)\rangle$, and suchlike, are computed through (30) via numerical estimation. An investor has made a profit if $\langle \Lambda_\ell(i,j)\rangle > 0$ and a loss otherwise. Those expectation values fully account for quantum fluctuations or, in terms of the interpretation of the model, for the trading activity in the lattice market model, including all possible arbitrage opportunities.

Under certain conditions the action (23) is invariant with respect to permutations of the asset accounts $i = 1, \ldots, m$. This can be seen by inspection of (24) and (25) making the ladder geometry explicit. We have

$$S_0 = \frac{1}{2} \sum_{j=0}^{n-1} \sum_{i=1}^{m} [P_{i0}(0,j) + P_{0i}(0,j) - 2] \tag{40}$$

and, reproducing only the first line of (25) for simplicity,

$$S_1 = \sum_{j=0}^{n} \sum_{i=1}^{m} d^+_{space} \bar{\Phi}(0,j)\Theta_i(0,j)\Phi(i,j) + \sum_{j=0}^{n-1} \sum_{i=1}^{m} d^+_{time} \bar{\Phi}(i,j)\Theta_0(i,j)\Phi(i,j+1)$$
$$+ \sum_{j=0}^{n-1} d^+_{axis} \bar{\Phi}(0,j)\Theta_0(0,j)\Phi(0,j+1) \ldots.$$

The couplings d^+_μ have been replaced by common values labeled by subscripts according to the discussion around Fig. 1, implying that those are independent of i. As this is our choice in the current work we arrive at permutation invariance with regard to the asset accounts.

For a given holding time interval ℓ, the time extent n of the lattice accommodates $n - \ell + 1$ operators (38). Hence, for a fixed ℓ, we have available $m(n - \ell + 1)$ measurements of $\langle \Lambda_\ell(i,j)\rangle$. Writing

$$\mathcal{S}_\ell = \{\langle \Lambda_\ell(i,j)\rangle \mid 1 \le i \le m,\ \ell \le j \le n\} \tag{41}$$

for these sets, where $1 \le \ell \le n$, we shall take their elements as instances of a stochastic variable and use it to estimate the probability distribution of profits (losses) in the lattice model market.

We wish to compare the results of the simulation to historical market data. Towards this end, there are two more considerations.

The first one is purely technical. Due to the finite time extent of the lattice, the size of the set $\mathcal{S}_\ell$ shrinks with increasing ℓ. We avoid this undesirable ℓ-dependence of the sample size by limiting the range of holding intervals to $1 \le \ell \le \ell_\infty$, where our choice is $\ell_\infty = \lfloor n/2 \rfloor$. Then we utilize only the terminal times slices j (see (38)) within $\ell_\infty \le j \le n$. Thus, the sets

$$\mathcal{S}_\ell^\infty = \{\langle \Lambda_\ell(i,j)\rangle \mid 1 \le i \le m,\ \ell_\infty \le j \le n\}, \tag{42}$$

with $1 \le \ell \le \ell_\infty$, all have the same number $m(n-\ell_\infty+1)$ of elements.

Secondly, historical data are commonly available as prices $P(t)$ at certain times t. Specifically, take those to be the average price of all instruments traded at a certain market. One might be tempted to compare $\log[P(t)/P(t')]$ to our lattice generated data. We argue that direct comparison is invalid because the instruments traded at times t and t' by various investors were being held for varying amounts of time by the market participants. The (published) historical price $P(t)$, being a cumulative measure determined by the supply–demand situation in the market, does not resolve this effect. On the other hand, our lattice data deliver price ratios at fixed holding time intervals ℓ, in some unit. In the present form the action (23) of the model does not dynamically generate the above effect. For the time being, to match the market situation, we consider the stochastic variable

$$X_L \in \bigcup_{\ell=1}^{\ell_\infty} w_\ell \otimes \mathcal{S}_\ell^\infty. \tag{43}$$

The notation used in (43) means that random numbers are drawn from the sets $\mathcal{S}_\ell^\infty$ with probability proportional to the weights w_ℓ. In practice, this can be accomplished by accordingly oversampling the lattice generated sets (42), for example. The stochastic variable X_L describes the distribution of the weighted average over gains realized at time j (the present) on instruments i that were bought at various times in the past. A possible choice for the weights is

$$w_\ell = \ell^{-\nu}, \tag{44}$$

where ν is a parameter. The specific mathematical form (44) of the weight factor has been chosen because a gauge theory is devoid of a scale. The only function compatible with this is a power law. The case $\nu > 0$ reflects a scenario in which fewer investors hold financial instruments for longer periods of time.

The scaling behavior described by the power law parameter in the context of our model is unique. Similar market behavior has been explored through the application of the fractional market hypothesis driven by quasi-rational traders [13]. With respect to the latter model, the gauge theory structure is identical to that of Ref. [6], where the gauge theory of arbitrage is strictly enforced in the classical sense. The resulting theoretical marketplace is analyzed through the eyes of quasi-rational investors. The traders interact through a series of hopping elements which are dependent upon a decision matrix composed of transition probabilities. These transition probabilities are a direct function of the predefined characteristic trading time of the investor which they represent. In Ref. [13], the investment horizon is applied directly to the asset price, thus fixing the characteristic time scale that the quasi-rational traders reflect upon. As a result of this application, the characteristic trading time defines the interaction of the various investors with varying investment horizons.

This characteristic trading time bears a resemblance to our lattice parameter ℓ. However, in the context of our model, ℓ represents investment horizons where the corresponding returns are extracted from the same market environment (regardless of the choice made for the characteristic trading time). In our lattice model, ℓ does not contribute to the asset–asset interactions. The interactions of our market are defined through the lattice parameters, which are executed through the updating algorithm's transition probability. This lattice interaction produces fixed asset prices, devoid of a scale, which are derived from gauge invariant quantities. The investor holding time ℓ is enforced through these gauge invariant constructs. The financial interpretation of the weight factor (44) applied to the investment horizon is similar to the scaling methods in Ref. [13]. The mathematical structure and algorithmic implications differ greatly between the two methods due to the structure and application of the two different characteristic trading times. Hence, direct comparison of the power law parameters' numerical values is not straightforward.

A set of historical data may be considered instances of a stochastic variable. This is a standard approach in mathematical finance [14]. We here consider

$$X_H \in \{\log(P(t)/P(t'))\}, \tag{45}$$

where the prices are from the NASDAQ index [15]. The sampling time interval $\tau = t - t'$ is one minute. We use data from 26 Aug. 2005 to 09 May 2007. The size of the set is 181 523. A histogram of the variable X_H for $\tau = 1$ min is shown in Fig. 3. There are 59 bins of width $\Delta X_H = 2.034 \times 10^{-4}$.

The numerical simulation has been done on a lattice with $n = 260$ time steps and $m = 30$ assets at $\beta = 4.0$ (see (26) and (30)), with 600 field configurations. The coupling constants in (25) are all equal, with $d_\mu^\pm = \bar{d}_\mu^\pm = 0.25$. All these parameters were *ad hoc* choices; no particular effort to tune them has been made.

The histogram in Fig. 3 displays the number of counts ΔN in each bin divided by the bin width ΔX. These approximate an unnormalized probability density function $p(X)$. We distinguish between the historical and the lattice results with subscripts H and L, and write

$$\frac{\mathrm{d}N_H}{\mathrm{d}X_H} = p_H(X_H) \quad \text{and} \quad \frac{\mathrm{d}N_L}{\mathrm{d}X_L} = p_L(X_L) \tag{46}$$

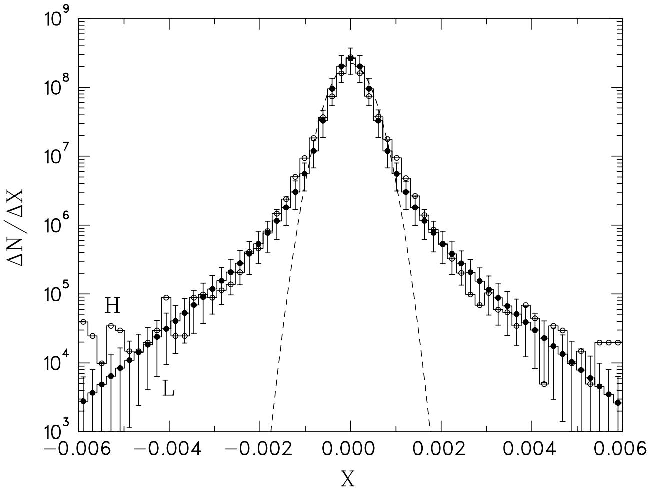

Fig. 3. Comparison of the relative gains distributions $\Delta N/\Delta X$ of the NASDAQ index of one-minute-interval historical data (H open circles) (45), and the corresponding results of the lattice simulation (L, filled circles) (43). The error bars are the standard deviations derived from 600 lattice configurations. The dashed line represents a fit to the lattice data with a Gaussian (normal) distribution.

for the respective distributions of X_H and X_L; see (45) and (43). Comparison of the distributions, $p_H \sim p_L$, proceeds by applying appropriate scale factors to the lattice generated data. Hence, writing

$$X_H = \lambda X_L \tag{47}$$

we aim at

$$p_H(X_H) \simeq \mu\, p_L(X_L), \tag{48}$$

provided appropriate values for λ and μ can be found. We expect λ to depend on the lattice parameters, including β, and μ is just a normalization constant. The lattice model results are compared to the historical data in Fig. 3. A good match is obtained with $\lambda = 6.0 \times 10^{-4}$ and $\nu = 2.01$ for the power law parameter in (44).

At this time it is important to point out that the two distributions are computed in slightly different ways. The lattice generated distribution displays returns calculated for varying holding period lengths, with the weights a function of the amount of time for which the assets are held by the various investors. The NASDAQ generated distribution displays returns calculated as the ratio of index prices collected at the one-minute frequency, where the prices reflect supply and demand interactions resulting from investors holding assets for varying lengths of time. However, a good match is nevertheless obtained through the fitting of the λ parameter that scales the magnitude of the lattice-driven returns and of the ν parameter that drives the implied average length of time for which assets are held by investors.

Remarkably, the agreement of the lattice model with historical data is valid through about four orders of magnitude. This is a considerable improvement over the original work by Mantegna and Stanley [7]. These authors matched high frequency historical data over less than two orders of magnitude by invoking a (truncated) stable Lévy flight probability distribution. The latter is not based on first principles though; in fact, it is an *ad hoc* assumption.

The dashed line in Fig. 3 represents a fit with a normal (Gaussian) distribution matched to data points in all bins of the lattice data. Equal normalization has been imposed as a constraint. Clearly both the historical and the lattice data are distinctly larger than the normal distribution. This observation has been referred to as fat tails [7]. These tails have consequences for certain results of financial modeling [16]. We observe that the present analysis reproduces fat tails with ease. It should be duly noted that the power law parameter ν determines the extent and steepness of the tails. We interpret this as saying that the presence of fat tails is a manifestation of the fact that investors hold their assets for varying amounts of time ℓ, with the instantaneous price being some combination of prices. This could be considered a novel insight into financial market dynamics. The historical data prove to be remarkably consistent with the lattice simulation, and the holding time modeling. This suggests that the gauge principle is in fact realized, to some degree of approximation, in real world financial market dynamics.

It is worth pointing out that one can also obtain a probability distribution curve displaying a certain amount of kurtosis (fat tails) with an exogenous specification of asset returns. Models allowing for stochastic volatility and/or jumps in the asset price are also able to produce fat tails and skewness in the distribution of the generated returns, as shown for instance in Bakshi et al. [17], among many others. The intuition in the case of stochastic volatility is that if the volatility is allowed to increase or decrease for a while, this will result in a larger-than-in-the-Gaussian-case probability mass in the tails and around the origin of the distribution (a phenomenon observed in most financial time series). Moreover, by allowing the stochastic volatility shocks to be negatively correlated with the asset returns shocks, one can fine-tune the level of skewness in the distribution of returns in addition to modeling the leverage effect whereby the financial market's

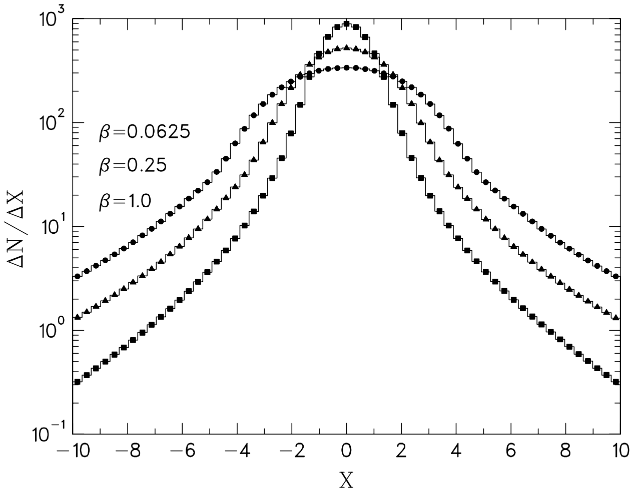

Fig. 4. Relative (unscaled) gains distributions $\Delta N/\Delta X$ of the lattice simulation (43) for varying values of β. Error bars are omitted for clarity; they are similar to those in Fig. 3.

volatility is often empirically observed to go up after a decline in the asset price. The intuition in the case of jumps is even more straightforward, since enabling the asset returns or prices to jump up or down independently of their otherwise Gaussian distribution will place one more often in the tails of the resulting probability curve. Alternatively, one can also embed ARCH/GARCH effects (Autoregressive Conditional Heteroskedasticity and Generalized Autoregressive Conditional Heteroskedasticity, respectively) in the very specification of the return-generating process [18]. Such specifications would imply, on average, that a large shock tends to be followed by a large shock, thus explaining the fat tails. Similarly, a small shock tends to be followed by a small shock, explaining the higher-than-Gaussian probability mass around the origin. However, all these models are exogenously specified, precisely engineered to yield returns distributions consistent with empirically observed ones. Our model, on the other hand, is a microscopic one relying solely on the principle of gauge invariance but is nevertheless able to deliver comparable results and match a high frequency distribution of market returns, without having pre-planted seeds of kurtosis into the model as in the case of stochastic volatility, jumps, and ARCH/GARCH setups.

The significance of the gauge model parameters with respect to financial observables warrants further discussion. This is a large area of unexplored territory to be addressed in future work. However, we here briefly explore the effect of changing the parameter β which defines the level of quantum fluctuations (arbitrage opportunities) present within the simulation. Fig. 4 displays the unscaled returns distribution for varying values of β where all other parameters are identical to that of Fig. 3. It is apparent that β characterizes the amount of returns that fall into the Gaussian regime, thus defining the quantity of the returns that contribute to the distributions' fat tails. Aside from rescaling, Fig. 4 demonstrates the stability of the distribution's characteristic shape with respect to varying β.

6. Outlook

The idea of scale invariance playing an important role in financial market dynamics has, of course, a long history [7]. An interesting heuristic approach is based on the assumption that the underlying dynamics will produce a self-organized critical system, which is well known to exhibit scale invariance [19]. An empirical analysis of historical data, using wavelet filtering to eliminate Gaussian noise, appears to confirm this hypothesis [20]. In the same vein, but somewhat closer to a microscopic description, scale-free network models [21] have been employed [22]. Here, the topology of link connections between traders is constructed to obey a certain power law from the outset. Our lattice study is remotely related to those ideas in the sense that self-organization could eventually be realized in modified versions of the lattice model.

Another known characteristic of market returns is the notion of volatility clusters, i.e. the fact that volatility may be high for some period of time and later remain low for some other period of time. One way to model and test this phenomenon is through the use of an ARCH model, developed by Engle [18]. In this model, the mean-corrected asset return r_t is serially uncorrelated but there is dependence through a quadratic function of its lagged values. A typical ARCH(m) model will assume that $r_t = \sigma_t \epsilon_t$ and $\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-2}^2 + \cdots + \alpha_m r_{t-m}^2$ with $\alpha_0 > 0$, and $\alpha_i \geq 0$ for $i > 0$, and where ϵ_t is independently and identically normally distributed with a mean of 0 and a variance of 1. Under this setup, large past shocks imply a large conditional variance for the next mean-corrected return. Interestingly, the empirical analysis of Ref. [20] seems to hint at such a scenario.

We are looking forward to investigating market characteristics from a microscopic point of view with future versions of the lattice gauge model. For example, features that we plan to study in the future involve correlations of assets over time, and the volatilities of these assets both in a stand-alone framework and in the context of a portfolio where several assets are pooled and interact. However, this will require the breaking of the time invariance principle currently in effect in our model, and will thus be a major undertaking.

Another area of investigation could be the phase structure of the lattice fields. In particular, the action could be modified to include larger, gauge invariant, building blocks such as loops and links extending over more than one (temporal) step.

7. Conclusion

We have implemented a proposal by Ilinski, that financial market dynamics should be independent of their arbitrary units used for cash and assets, thus being describable using a local gauge theory which, in some sense, resembles quantum electrodynamics. The model relies on the dilation group $\mathbb{R}^+$ as a local gauge symmetry on a lattice multi-ladder geometry of dimension $m + 1$, where m is the number of investment instruments coupled to a cash account, and the remaining degree of freedom represents the time evolution of the system. The numerical simulation of the model yields the probability distribution of a stochastic variable which represents the market average of relative gains of investments. It is matched to high frequency historical data of the NASDAQ index and found to reproduce these over four orders of magnitude; in particular, the empirical observation of fat tails, meaning those that exceed a normal distribution, is reproduced. In this model, the microscopic reason for this observation is traced back to the fact that a spot price of an asset is a result of the investing history: The assets offered at a particular time have been held for various amounts of time.

We side with Ilinski [6] that the gauge principle is a very important feature of market dynamics, though it may not be the only one. In particular, terms correcting the lattice action, which break gauge invariance (one hopes, perturbatively), could be added to the model. These terms may account for inflation rates, dividend growth, and the like. Also, one could implement boundary conditions on the lattice model, for example the historical Federal Reserve interest rate, or the time history of specific company stock values. This is a particularly interesting avenue because of the potential of making predictions of future gains distributions. Of course, future prices cannot be predicted, but the time evolution of their probability distribution can be, at least in the short term.

We close with the remark that the present model is only an, initial, exploratory attempt. No exploration of the lattice features with regard to the parameters has been made. This would include the search for phase transitions, or spontaneous symmetry breaking, for example. We shall leave this to future work.

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