Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER Journal of Economic Dynamics & Control 31 (2007) 887–905
JOURNAL OF Economic Dynamics & Control
www.elsevier.com/locate/jedc

The impact of fat tails on equilibrium rates of return and term premia ☆

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Received 22 February 2006; accepted 8 March 2006
Available online 25 April 2006

Abstract

We investigate the impact of ignoring fat tails observed in the empirical distributions of macroeconomic time series on the equilibrium implications of the consumption-based asset-pricing model with habit formation. Fat tails in the empirical distributions of consumption growth rates are modeled as a dampened power law process that nevertheless guarantees finiteness of moments of all orders. This renders model-implied mean equilibrium rates of return and equity and term premia finite. Comparison with a benchmark Gaussian process reveals that accounting for fat tails lowers the model-implied mean risk-free rate by 20 percent, raises the mean equity premium by 80 percent and the term premium by 20 percent, bringing the model implications closer to their empirically observed

☆An earlier version of this paper authored by Prasad Bidarkota was circulated under the title, "On the Economic Value of Modeling Fat Tails: Measuring the Impact on Equilibrium Risk Premiums". We thank participants at the 2004 Quantitative Methods in Finance conference (UTS, Sydney) and two anonymous referees for comments and suggestions. Any errors are our own.

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0165-1889/$ - see front matter
doi:10.1016/j.jedc.2006.03.001

counterparts. Fat tails also increase the model-implied volatility of the risk-free rate and the equity premium.

JEL classification: G12; G13; E43

Keywords: Asset-pricing model; Habit formation; Term premium; Equity premium; Fat tails; Dampened power law

1. Introduction

The empirical distributions of many economic and financial time series exhibit fat tails. This has been well documented in the literature.[1] The presence of fat tails warrants the use of probability distributions that can accommodate the likelihood of large positive or negative shocks impacting the economy. Gaussian distributions do not admit this possibility. However, these distributions are well understood and analytically tractable. This explains their pervasive use across macroeconomics and finance.

Non-Gaussian fat-tailed distributions have been used to some extent, especially in models of asset pricing. In such studies, the underlying asset price is typically assumed to either follow a Gaussian distribution supplemented by features such as stochastic volatility and/or jumps (thus making the resulting distribution non-Gaussian), or simply assumed to follow a non-Gaussian distribution such as an α-stable distribution. However, one reason why departures from a Gaussian distribution are not more common, despite its documented deficiencies, is the added complexity concomitant with such departures. Also, often in economic models, the use of many non-Gaussian distributions precludes the possibility of finding exact analytical solutions to equilibrium quantities of interest.

One may then ask what the cost of ignoring fat tails in economic model building would be.[2] One answer is available from the option pricing literature where the assumption of normality is relaxed to accommodate the possibility of large movements in prices of underlying assets. In this vein, Hales (1997) finds that an options pricing model with α-stable distributions that capture fat tails due to McCulloch (1987) reduces pricing biases relative to the Gaussian Black and Scholes (1973) model for valuing foreign currency options. A second answer is available from the asset allocation literature. Here, Tokat et al. (2003) find that the optimal allocation of wealth between risk-free and risky assets could be up to 26 percent different when one accounts for fat tails in the empirical distributions of underlying data. In an asset-pricing context, Bidarkota and McCulloch (2003) find that

[1]Studies documenting fat tails in macroeconomic series include, among others, Blanchard and Watson (1986), Balke and Fomby (1994), and Kiani and Bidarkota (2004).

[2]Outside the context of an explicit economic model, the danger inherent in ignoring fat tails in empirical distributions in economic analysis is illustrated in studies such as those on value-at-risk measures (Khindanova et al., 2001).

accounting for fat tails in the dividends data generates an additional 13 percent of equilibrium equity returns in the standard consumption-based asset-pricing model.

In this study we seek to provide another answer to the question on the economic costs of ignoring fat tails in economic models. We provide an answer in the context of the popularly used consumption-based asset-pricing model of Lucas (1978), augmented with a habit-formation feature as in Abel (1999). We study two versions of the model – one in which exogenous consumption stochastically evolves as a non-Gaussian process exhibiting fat tails and the benchmark version in which consumption evolves as a Gaussian process. We calculate the model-implied equilibrium rates of return, the equity and the term premia in the two versions and compare them to evaluate the economic impact of modeling fat tails.

We model fat tails in this paper with the dampened power law (DPL), recently utilized by Wu (2006) to examine the tail behavior of financial security returns and option prices. DPL nests α-stable distributions but, unlike these, has the advantage that all moments are finite under strictly positive dampening. This renders model-implied rates of return and equity and term premia finite under certain restrictions.

The paper is organized as follows. We set out the asset-pricing model in Section 2. In Section 3, we specify the stochastic process that accounts for fat tails, discuss an estimation method, and then specialize solutions to equilibrium quantities of interest implied by the asset-pricing model to the postulated stochastic structure. In Section 4, we report maximum likelihood estimates of the model with data from the US, and calculate the model-implied equilibrium rates of return and equity and term premia. In Section 5, we conclude with the main observations derived from our study.

2. The asset-pricing model

In this section we provide a description of the asset-pricing model due to Abel (1999) that forms the basis for our study. We specify preferences, define a canonical asset, note the stochastic structure, outline the key steps for solving for equilibrium asset prices, and define the rate of return on the canonical asset and the term premium. The content of this section is largely derived from Abel (1999).

2.1. Preferences

The model economy is populated by a continuum of identical infinitely-lived agents. It is a closed economy, producing a single completely perishable output. Thus, consumption in every period must equal output.

A representative consumer maximizes expected lifetime utility given by

$$U_t = \mathrm{E}_t\left\{\sum_{j=0}^{\infty}\frac{1}{(1+\delta)^j}u\left(c_{t+j}, v_{t+j}\right)\right\}, \tag{1}$$

where

$$u(c_t, v_t) = \frac{1}{1-r}\left(\frac{c_t}{v_t}\right)^{1-r}, \quad r>0, \quad \delta>0. \tag{2}$$

Here, c_t is an individual agent's consumption in period t, v_t is a benchmark level of consumption assumed exogenous to the individual consumer, and r is the coefficient of relative risk aversion (CRRA).

The benchmark level of consumption is assumed to depend on the *aggregate* per capita level of consumption C_t as follows:

$$v_t \equiv C_t^{h_0} C_{t-1}^{h_1} \left(G^t\right)^{h_2}, \tag{3}$$

where $G \geqslant 1$ and $0 \leqslant h_i \leqslant 1$ for $i = 0, 1, 2$. Setting $h_1 > 0$ and $h_0 = h_2 = 0$ produces the 'catching up with the Joneses' utility specification of Abel (1990).

The intertemporal marginal rate of substitution (IMRS) between period t and period $t+1$ is given by

$$M_{t+1} \equiv \frac{1}{1+\delta} \frac{u_c(c_{t+1}, v_{t+1})}{u_c(c_t, v_t)}, \tag{4}$$

Using Eqs. (2) and (3) and recognizing that, in equilibrium,

$$x_{t+1} \equiv \frac{C_{t+1}}{C_t} = \frac{c_{t+1}}{c_t} \tag{5}$$

the IMRS can be written as:

$$M_{t+1} = \beta x_{t+1}^{-A} x_t^{\theta}, \tag{6a}$$

where

$$\beta \equiv \frac{1}{1+\delta} G^{h_2(r-1)} > 0 \tag{6b}$$

$$A \equiv r(1-h_0) + h_0 > 0 \tag{6c}$$

and

$$\theta \equiv h_1(r-1). \tag{6d}$$

The equilibrium asset prices and returns depend on the IMRS.

In the model, there are six preference parameters – δ, r, h_0, h_1, h_2, and G – and all six parameters determine the IMRS as is evident from the equations above. However, there are only three independent parameters – β, A, and θ – that determine the IRMS as given in Eq. (6a).

2.2. The Canonical Asset

Abel (1999) introduces a canonical asset that includes fixed income securities of all maturities and equities as special cases. The canonical asset is an n-period asset, with the current period indexed by t and terminal period by $t+n$. This asset pays $a_j y_{t+n-j}^{\lambda}$ in the period that is j periods before the terminal period for $j = 0, \ldots, n-1$, where

$y_{t+n-j}>0$ is a random variable, $a_0>0$ is a constant, and $a_j\geqslant 0$, $j=1,\ldots,n-1$ are constants. The parameter λ takes the value zero for fixed income securities and one for equities.

Thus, the payoff for fixed-income securities in period $t+n-j$ is the known amount a_j. In the Lucas (1978) fruit-tree model, the dividend (per capita) on equity equals consumption per capita C_t. In terms of the canonical asset, this equity can be represented with $n=\infty$, $a_j=1$ for all $j\geqslant 0$, and $y_t\equiv C_t$.

Let $p_t(n,\lambda)$ denote the ex-payment price of the canonical n-period asset in period t. The dependence of this price on the sequence of constants a_j, $j=0,\ldots,n-1$ and on the stochastic process for y_t is suppressed for notational convenience. The gross rate of return on the canonical asset between period t and period $t+1$ is given by

$$R_{t+1}(n,\lambda)\equiv\frac{p_{t+1}(n-1,\lambda)+a_{n-1}y_{t+1}^{\lambda}}{p_t(n,\lambda)},\quad \text{for } n\geqslant 1. \tag{7}$$

2.3. The Stochastic Structure

The payoff growth rates $z_{t+1}\equiv y_{t+1}/y_t$ and the consumption growth rates $x_{t+1}\equiv C_{t+1}/C_t$ observable at the beginning of period $t+1$ are assumed throughout this paper to follow i.i.d. processes.

2.4. Asset prices

Asset prices are postulated to be given by

$$p_t(n,\lambda)=\omega(n,\lambda)x_t^{\theta}y_t^{\lambda}, \tag{8}$$

where $\omega(n,\lambda)$ is a function to be determined. The first order condition for utility maximization in this model is as follows:

$$\mathrm{E}_t\{R_{t+1}(n,\lambda)M_{t+1}\}=1. \tag{9}$$

Substituting Eq. (8) in Eq. (7) and the resulting expression for the gross rate of return into the first order condition above yields the following difference equation in $\omega(n,\lambda)$ under the assumption that z_{t+1} and x_{t+1} follow i.i.d. processes:

$$\omega(n,\lambda)=\kappa(\lambda)\omega(n-1,\lambda)+\beta a_{n-1}\mathrm{E}\{x_{t+1}^{-A}z_{t+1}^{\lambda}\}, \tag{10a}$$

where

$$\kappa(\lambda)=\beta\mathrm{E}\{x_{t+1}^{\theta-A}z_{t+1}^{\lambda}\}. \tag{10b}$$

Throughout this paper, we assume as in Abel (1999) that $0<\kappa(\lambda)<1$. This assumption guarantees that the difference equation (10a) converges as n grows.

The fact that the price of a zero-period asset has to be zero, i.e. that $p_t(0,\lambda)=0$, provides the boundary condition for solving the difference Eq. (10a). Using this boundary condition and Eq. (8) in Eq. (10a) yields:

$$\omega(1,\lambda)=\beta a_0\mathrm{E}\{x_{t+1}^{-A}z_{t+1}^{\lambda}\}>0. \tag{11}$$

The solution to the difference equation with the above boundary condition, as can be easily verified, is given by

$$\omega(n,\lambda) = \frac{\omega(1,\lambda)}{a_0}\sum_{i=1}^{n} a_{i-1}[\kappa(\lambda)]^{n-i}. \tag{12}$$

2.5. Expected Rate of Return on the Canonical Asset

Starting from Eq. (7), the expected rate of return on the one-period canonical asset can be shown to be:

$$\mathrm{E}\{R_{t+1}(1,\lambda)\} = \frac{\mathrm{E}\{z_{t+1}^{\lambda}\}}{\beta \mathrm{E}\{x_{t+1}^{-A} z_{t+1}^{\lambda}\} x_t^{\theta}}. \tag{13}$$

The expected rate of return on an n-period canonical asset can then be shown to be:

$$\mathrm{E}\{R_{t+1}(n,\lambda)\} = \left[\Psi + (1-\Psi)\frac{a_{n-1}}{a_0}\frac{\omega(1,\lambda)}{\omega(n,\lambda)}\right]\mathrm{E}\{R_{t+1}(1,\lambda)\}, \tag{14a}$$

where

$$\Psi = \frac{\mathrm{E}\{x_{t+1}^{\theta} z_{t+1}^{\lambda}\}\mathrm{E}\{x_{t+1}^{-A} z_{t+1}^{\lambda}\}}{\mathrm{E}\{z_{t+1}^{\lambda}\}\mathrm{E}\{x_{t+1}^{\theta-A} z_{t+1}^{\lambda}\}}. \tag{14b}$$

2.6. Term Premia

The term premium on an n-period asset is defined as

$$TP(n,\lambda) \equiv \frac{\mathrm{E}\{R_{t+1}(n,\lambda)\}}{\mathrm{E}\{R_{t+1}(1,\lambda)\}} - 1. \tag{15}$$

Using Eq. (14a), this becomes

$$TP(n,\lambda) = (\Psi - 1)\Upsilon(n,\lambda), \tag{16a}$$

where

$$\Upsilon(n,\lambda) \equiv 1 - \frac{a_{n-1}}{a_0}\frac{\omega(1,\lambda)}{\omega(n,\lambda)} = 1 - \frac{a_{n-1}}{\sum_{i=1}^{n} a_{i-1}[\kappa(\lambda)]^{n-i}}. \tag{16b}$$

The term $(\Psi - 1)$ is the term premium scale factor; it does not depend on the maturity n. It can be easily seen from Eq. (14b) that when $\theta = 0$, $\Psi = 1$ and hence the term premium scale factor is zero. This happens when either utility is logarithmic ($r = 1$ or when $h_1 = 0$ from Eq. (6d)). For an n-period discount bond, $a_1 = \cdots = a_{n-1} = 0$. Therefore, from Eq. (16b), $\Upsilon(n,0) = 1$ for $n > 1$. Thus, the term premium is independent of the maturity n for pure discount bonds with more than one period to run.

3. Exogenous driving process and equilibrium rates of return and term premia

In Section 3.1 we define explicitly the dampened power law process for consumption growth rates, and in Section 3.2 discuss estimation of the process. In Section 3.3 we specialize the formulae for the term premium and the expected rates of return on the canonical asset to risk-free discount bonds and equity that pays consumption goods as in the Lucas (1978) fruit tree model. In Section 3.4 we outline the benchmark Gaussian consumption process.

3.1. Stochastic Process for Consumption Growth Rates – Dampened Power Law

Economic and financial asset returns have been shown to possess distributions displaying power law decay in their tails, indicating that the tails are thicker than what one would find in the Gaussian case. These fat-tailed distributions are often said to have 'power tails' that are inconsistent with the common Gaussian distributional assumption. However, most asset returns also converge to a Gaussian distribution when aggregated over time. This fact is inconsistent with the assumption of an α-stable distribution as a possible explanation for the observed power tails mentioned above, since time-aggregation of stable distributions yields a stable distribution. In a recent article, Wu (2006) focuses on reconciling these apparently contradicting observations by modeling asset returns with a DPL.

The DPL model aims at reproducing the power tails observed in the finance and economics literatures, simultaneously allowing time aggregation to lead to Gaussian distributions (by permitting the Central Limit Theorem to hold). This is accomplished by the dampening of the tails of the probability distributions with an exponential function that nevertheless permits accurate modeling of the power tail distributions observed empirically. Dampening also guarantees the existence of finite moments of all orders. Without dampening, not all moments of the power tail distributions are finite and hence time aggregation would not necessarily lead to Gaussian behavior.

In the DPL setting, we assume that the consumption growth rates follow:

$$\ln(x_{t+1}) = \mu + \varepsilon_{t+1}, \tag{17a}$$

where ε_t is a pure jump Lévy process following a DPL, with its Lévy density – controlling the distribution – defined by

$$v(\varepsilon) = \begin{cases} \gamma_+ \exp\{-\beta_+|\varepsilon|\}|\varepsilon|^{-\alpha-1}, & \varepsilon > 0, \\ \gamma_- \exp\{-\beta_-|\varepsilon|\}|\varepsilon|^{-\alpha-1}, & \varepsilon < 0, \end{cases} \tag{17b}$$

where $\alpha \in (0,2]$ and $\gamma_+, \gamma_-, \beta_+, \beta_- > 0$. The β parameters control the amount of dampening, whereas the γ parameters determine the symmetry of the distribution. The α parameter is identical to the α parameter found in the traditional α-stable distributions, guiding the amount of leptokurtosis in the tails.

We adopt this DPL process for describing the exogenously evolving consumption growth rates in our paper because it allows simultaneously for fat tails and the existence of finite moments (see Geweke, 2001 for an elaboration on this point).

3.2. Estimation Issues

Before estimating the parameters associated with the Lévy process, we first estimate the mean growth rate of consumption and use it to demean the series. Note that properties of logarithms imply that computing a simple arithmetic average of log differences of consumption is essentially the same as averaging the first and last observation, which can yield a noisy estimate of the mean growth rate. Hence, following Wu (2006), we regress log consumption levels on time t instead, estimating

$$\ln C_t = a + bt + u_t. \tag{18}$$

From the discrete setting of Eq. (17a), the estimate for b is thus an estimate of the mean annualized growth rate of consumption μ. We use this estimate of μ to detrend the consumption series, and model the log-difference of detrended consumption data as a pure-jump Levy DPL process.

The cumulant exponent of the Lévy DPL process given in Section 3.1 above is derived in Wu (2006):

$$k(s) = \Gamma(-\alpha)\gamma_+[(\beta_+ - s)^\alpha - \beta_+^\alpha] + \Gamma(-\alpha)\gamma_-[(\beta_- + s)^\alpha - \beta_-^\alpha] + sQ, \tag{19}$$

where Γ represents the gamma function defined by

$$\Gamma(x) = \int_0^\infty \mathrm{t}^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t. \tag{20}$$

and where Q is an immaterial constant that gets cancelled out once the residual u_t is adjusted so that it has zero mean (see Wu, 2006 for details).

Since the cumulant exponent is defined as

$$k(s) = \frac{1}{t}\log\,\mathrm{E}[\exp\{s\varepsilon_t\}], \tag{21}$$

we can use the fact that $\mathrm{E}[\exp\{s\varepsilon_t\}] = \exp\{t\,k(s)\}$ along with the expression for $k(s)$ from Eq. (19) in order to compute exponential moments of the DPL process. These exponential moments are directly utilized in computing the equilibrium expected riskless rate of return, the equity premium and the term premium in the empirical section of the paper.

The characteristic function of the DPL process for an interval of time t is

$$\Phi_{\varepsilon_t}(s) = \mathrm{E}[\exp\{\mathrm{i}s\varepsilon_t\}] = tk(is). \tag{22}$$

We can thus use the expression for $k(s)$ in order to recover the probability density function of the Lévy density by standard Fourier inversion transformation. Once the density function is retrieved, the parameters associated with the DPL can be estimated by maximum likelihood.

3.3. Equilibrium Rates of Return and Term Premia

In this section we specialize the formulae for the expected rates of return on the canonical asset and the term premium defined in Sections 2.5 and 2.6 to risk-free discount bonds and equity that pays consumption goods as in the Lucas (1978) fruit tree model. For this kind of equity, the stochastic payoff equals consumption per capita so that $y_t \equiv C_t$, and hence the payoff growth rates equal consumption growth rates so that $z_{t+1} \equiv x_{t+1}$ in the notation introduced in Section 2.3.

Using the above identities, starting from Eq. (13), the unconditional mean of the riskless rate on a discount bond for which $\lambda = 0$ can be shown to be

$$\mathrm{E}\{R_{t+1}(1,0)\} = \frac{\mathrm{E}\{x_t^{-\theta}\}}{\beta\mathrm{E}\{x_{t+1}^{-A}\}}. \tag{23}$$

Similarly, starting from Eq. (14a), the unconditional mean of equity returns for which $\lambda = 1$ can be shown to be

$$\mathrm{E}\{R_{t+1}(\infty,1)\} = [1 + \kappa(1)(\Psi - 1)]\mathrm{E}\{R_{t+1}(1,1)\}, \tag{24a}$$

where

$$\mathrm{E}\{R_{t+1}(1,1)\} = \left[\frac{\mathrm{E}\{x_{t+1}\}\mathrm{E}\{x_{t+1}^{-A}\}}{\mathrm{E}\{x_{t+1}^{1-A}\}}\right]\mathrm{E}\{R_{t+1}(1,0)\}, \tag{24b}$$

and from Eq. (10b),

$$\kappa(1) = \beta\mathrm{E}\{x_{t+1}^{1+\theta-A}\}. \tag{25}$$

From Eq. (14b), for discount bonds with $\lambda = 0$,

$$\Psi = \frac{\mathrm{E}\{x_{t+1}^{\theta}\}\mathrm{E}\{x_{t+1}^{-A}\}}{\mathrm{E}\{x_{t+1}^{\theta-A}\}}. \tag{26}$$

As indicated in Section 2.6, the term premium on discount bonds is just the term premium scale factor $(\psi-1)$.

Since the expressions for the mean risk-free rate, mean equity premium and term premium of Eq. (23)–(26) involve terms such as $\mathrm{E}\{x^{A}\}$ or $\mathrm{E}\{x^{\theta-A}\}$, we can compute these expressions using Wu's (2006) cumulant exponent formula given in Eq. (19). Wu (2006) shows in his Proposition 2 that with $\gamma_+,\gamma_-,\beta_+,\beta_- > 0$, the cumulant exponent is well defined only for $s \in (-\beta_-,\beta_+)$. Thus, we need to ensure that Wu's Proposition 2 is not violated when computing equilibrium mean rates of return and the term premium. Examining the expressions that need to be computed above, and similar expressions for computing volatility of rates of return (not reported), we need the following terms to lie in the interval $(-\beta_-,\beta_+)$: $\theta-A$, $1+\theta-A$, θ, $-A$, $1+\theta$, $1-A$, $1,-\theta$, 2, -2θ, $2(\theta+1)$. This imposes restrictions on the range of values that one can entertain for the preference parameters including the relative risk aversion coefficient used subsequently in calculating the model implied rates of return and the term premium.

3.4. The Log-Normal Case

In the benchmark log-normal case, we assume that the consumption growth rates follow an i.i.d. Gaussian process:

$$\ln(x_{t+1}) = \mu + \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim \text{iidN}(0, \sigma^2). \tag{27}$$

Given the moment generating function of the normal distribution, we can readily evaluate analytically all the expressions given in Section 3.3 for the model-implied equilibrium quantities of interest when the consumption growth rates x_{t+1} (which equal the payoff growth rates z_{t+1}) are assumed to be i.i.d. Gaussian. We present these formulae below.

From Eq. (23), the expected gross rate of return on a discount bond equals

$$\mathrm{E}\{R_{t+1}(1,0)\} = \beta^{-1}\exp\left\{(A-\theta)\mu - \frac{1}{2}\left(A^2-\theta^2\right)\sigma^2\right\}. \tag{28}$$

From Eq. (24a), the expected gross rate of return on equity becomes

$$\mathrm{E}\{R_{t+1}(\infty,1)\} = \beta^{-1}\left[1+\kappa(1)\left\{\exp\left(\theta A\sigma^2\right)-1\right\}\right]\exp\left\{A\sigma^2\right\}\exp\left\{(A-\theta)\mu - \frac{1}{2}\left(A^2-\theta^2\right)\sigma^2\right\}, \tag{29}$$

where from Eq. (25),

$$\kappa(1) = \beta\exp\left\{(1+\theta-A)\mu + \frac{1}{2}(1+\theta-A)^2\sigma^2\right\}. \tag{30}$$

From Eq. (26),

$$\Psi = \exp\left\{\theta A\sigma^2\right\}. \tag{31}$$

From Eq. (16a) and the discussion that follows, the term premium on riskless discount bonds becomes

$$TP(n,0) = \Psi - 1. \tag{32}$$

4. Evaluating model-implied premia and expected rates of return

In Section 4.1 we discuss the consumption data series used and report summary statistics. In Section 4.2 we report the maximum likelihood estimates of the DPL process and the benchmark Gaussian process for the data. In Section 4.3 we compute the model-implied expected rates of return and the term premium and discuss the quantitative implications of modeling fat tails.

4.1. Characteristics of the Consumption Data

We use annual US real per capita consumption data on non-durables and services from Campbell and Cochrane (1999) spanning the period 1889–1997. Fig. 1 plots the

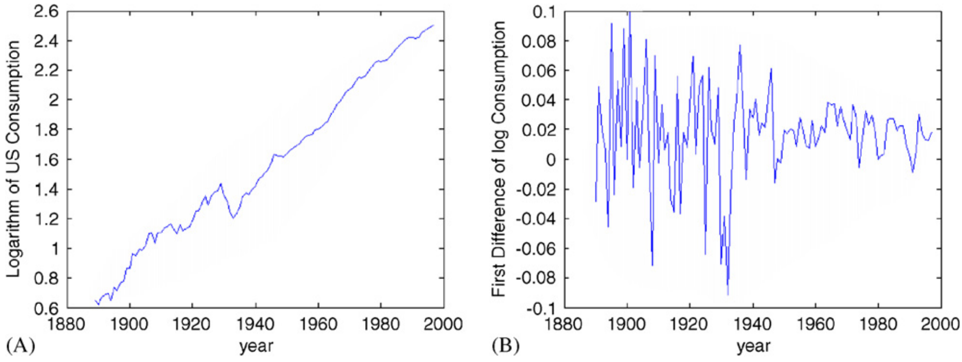

Fig. 1. Plots of real per capita consumption data.

Table 1
Summary statistics of real per capita consumption data

	Mean	Variance	Skewness	Kurtosis	Normality test
Real per capita Consumption growth rates	1.726e–2 (3.105e–3)	1.041e–3 (1.417e–4)	−0.503 (0.984)	4.555 (4.847e–4)	15.442 (4.434e–4)

Notes: Numbers in parentheses in the first two columns are the standard errors for the mean and variance. Numbers in parentheses in the third and fourth columns are the *p*-values for the null hypothesis of no skewness and no excess kurtosis, respectively. The normality test gives the Jarque-Bera test statistic and the *p*-value in parentheses.

real consumption data and Table 1 presents summary statistics. The mean growth rate is 1.726 percent per annum. Kurtosis is measured to be in excess of, and statistically significantly greater than, 3 indicating fat tails in the empirical histogram. Normality is strongly rejected by the Jarque–Bera test (*p*-value is 4.43e-4).

4.2. Model Estimates for the Consumption Growth Rates

Table 2 presents empirical results on maximum likelihood estimates of Eqs. (17) and (27). We estimate the most general unconstrained version of the DPL process presented in Eqs. (17) and several restricted versions.The first row reports estimation results for the most general version. The second row reports estimation results for the symmetric dampening case where the dampening parameters $\beta_+ = \beta_-$. The third row reports estimates obtained by fitting a DPL process without dampening that is identical to fitting an α-stable process to the consumption growth rates. The next row reports estimates obtained by fitting a symmetric α-stable process. The last row reports results of fitting the benchmark Gaussian model. Incremental benefits of the most general version of the DPL process can be measured by the log-likelihood values in the last column.

Table 2
Maximum likelihood model estimates.

Dampened Power Law Process $\ln(x_{t+1}) = \mu + \varepsilon_{t+1} \quad \varepsilon_{t+1} \sim iid\ \mathrm{DPL}(\alpha, \gamma_+, \gamma_-, \beta_+, \beta_-)$ (17a)

Gaussian Process $\ln(x_{t+1}) = \mu + \varepsilon_{t+1} \quad \varepsilon_{t+1} \sim iid\ \mathrm{N}(0, \sigma^2)$ (27)

	α	γ_+	γ_-	$\gamma_+ = \gamma_-$	β_+	β_-	$\beta_+ = \beta_-$	σ^2	Log L
Most general model	1.73	2.9e–4	3.4e–4		10.00	7.00			218.49
	(0.03)	(1.2e–4)	(1.2e–4)		(3.41)	(3.52)			
Symmetric dampening	1.74	2.3e–4	2.9e–4				4.00		216.67
	(0.01)	(1.32e–6)	(2.01e–7)				(0.03)		
No dampening	1.74	2.1e–4	2.6e–4		0	0			216.35
	(0.02)	(1.1e–4)	(1.2e–4)		(Restricted)	(Restricted)			
No dampening on	1.74			2.4e–4	0	0			215.92
Symmetric stables	(0.01)			(8.3e–5)	(Restricted)	(Restricted)			
Gaussian	2								
	(Restricted)							1.11e–3	213.92
								(1.5e–4)	

Notes: Numbers in parentheses are the standard errors.

The most general model yields estimates of 1.73 for α, scaling parameters $\gamma_+ = 0.00029$ and $\gamma_- = 0.00034$, and dampening parameters $\beta_+ = 10$ and $\beta_- = 7$. The difference in scaling parameters ($\gamma_+ - \gamma_-$) indicates a slight degree of negative skewness in the distribution of consumption growth rates. The dampening coefficients β_+ and β_- are both large, and statistically significantly positive, thus guaranteeing the existence of finite moments of all orders (see Wu's, 2006 Proposition 1). All other estimates are also statistically significant at the 0.05 level.

Estimates of the common parameters for the restricted nested models are generally similar to those for the unrestricted case reported above. In the symmetric dampening case the common dampening coefficient $\beta = 4$. Standard likelihood ratio (LR) test (not reported) would reject symmetric dampening in favor of the general model in Eq. (17) at the 0.10 significance level. Similarly an LR test would reject the Gaussian process in favor of the dampened power law process for consumption growth rates at the 0.10 significance level.

Armed with parameter estimates, rates of return and the equity and term premia implied by the model can now be computed alternatively under the DPL and Gaussian process for consumption growth rates. A comparison of these quantities would provide a quantitative assessment of the implications of modeling fat tails.

4.3. Implied Premia and Expected Rates of Return

As noted at the end of Section 2.1 our asset-pricing model has six preference parameters. We need to select values for each of these parameters before we can compute equilibrium quantities of interest implied by our model. We follow Abel (1999) in imposing the following three restrictions on these parameters: $h_0 = 0$, $h_0 + h_1 + h_2 = 1$, and $G = 1 + \mu$. The first restriction $h_0 = 0$ implies that the benchmark level of consumption v_t does not depend on the current period aggregate per capita level of consumption C_t as evident from Eq. (3). The third restriction $G = 1 + \mu$ captures the intuitive notion that the growth rate of the benchmark level of consumption over time reflects the growth rate of the aggregate per capita level of consumption.

Following the discussion in Section 3.2 related to estimation of the mean growth rate of aggregate consumption, we use an estimate of $\mu = 2.58$ percent/annum obtained by estimation of Eq. (18).

Following the model parameterization in Abel (1999), we choose $h_1 = 0.15$. We choose the rate of time preference δ to be 0.02. This leaves us with just one parameter to choose, namely, the coefficient of relative risk aversion r. We report results for different values of r below. Following the discussion at the end of Section 3.3, in order to ensure finiteness of exponential moments used in computing model-implied equilibrium rates of return and the term premium, the coefficient of relative risk aversion must be less than the estimate of β_-. Thus, using the estimate of β_- for the unrestricted DPL model from Section 4.2, r is constrained to be less than 7.

Table 3 presents the model-implied equilibrium expected risk-free rates, equity premium, and the term premium for the unrestricted DPL consumption growth rate process and the log-normal process. All rates of return and the term premium are

Table 3
Expected rates of return and term premia

		$r = 1.5$	$r = 2$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
Mean risk-free rate	DPL	4.540	4.449	4.161	3.749	3.205	2.521	1.677
$E\{R_{t+1}(1{,}0)\}$	Log-normal	4.550	4.464	4.208	3.839	3.359	2.770	2.074
	% Difference	−0.2	−0.4	−1.1	−2.3	−4.6	−9.0	−19.1
Mean equity premium	DPL	0.390	0.540	0.871	1.247	1.671	2.155	2.729
$E\{R_{t+1}(\infty{,}1)\}-E\{R_{t+1}(1{,}0)\}$	Log-normal	0.187	0.266	0.450	0.667	0.917	1.199	1.513
	% Difference	108.4	102.6	93.5	86.9	82.3	79.7	80.4
Term premium	DPL	0.014	0.036	0.110	0.222	0.376	0.577	0.842
$TP(n{,}0)$	Log-normal	0.012	0.033	0.100	0.200	0.334	0.501	0.702
	% Difference	8.4	8.8	9.8	11.1	12.8	15.3	20.0

Notes: All statistics are expressed in percent per annum. % Difference is the difference in the relevant statistic between the DPL and the Log-normal cases, relative to the value in the Log-normal case.

expressed in percent per annum. As is evident from the table, the asset-pricing model is able to generate a low enough mean risk-free rate of under 3 percent per annum with relative risk aversion coefficient of about 6. The mean equity premium for this CRRA coefficient is 2.2 percent per annum, which is higher than what Mehra and Prescott (1985) are able to generate without habit formation but still much lower compared to the historical level of about 7 percent per annum.

The model generates a term premium on risk-free bonds of about 0.6 percent per annum with a CRRA coefficient of 6. Abel (1999) reports a term premium on long term US government-issued fixed-income securities of 170 basis points per year. Abel (1999) has greater success in replicating both the empirically observed mean risk-free rate and the equity and term premium with the asset-pricing model used here by incorporating leverage in the model. Our main objective in this paper is to evaluate the quantitative importance of modeling fat tails on the model-implied equilibrium rates of return and the term premium in a relatively simple framework. We therefore did not entertain the possibility of leverage in the version of the asset-pricing model considered here.

Comparing the model-implied rates of return in the DPL and log-normal cases in Table 3, we find that accounting for fat tails leads to a lower mean risk-free rate, and higher mean equity and term premiums. More specifically, the DPL model is able to lower the mean risk-free rate by as much as 20 percent, raise the mean equity premium by 80 percent or more, and raise the term premium by 20 percent compared to the log-normal case. Thus, accounting for fat tails produces a closer match of the quantitative implications of the consumption-based asset-pricing model with the stylized facts observed in the macroeconomic and financial data.

Fig. 2 plots the model-implied equilibrium mean risk-free rate, the mean equity and term premiums as a function of the coefficient of relative risk aversion in both the DPL and the log-normal cases. As the graphs indicate, accounting for fat tails has greater impact on the implied rates of return and the equity and term premiums at higher values of the CRRA coefficient.

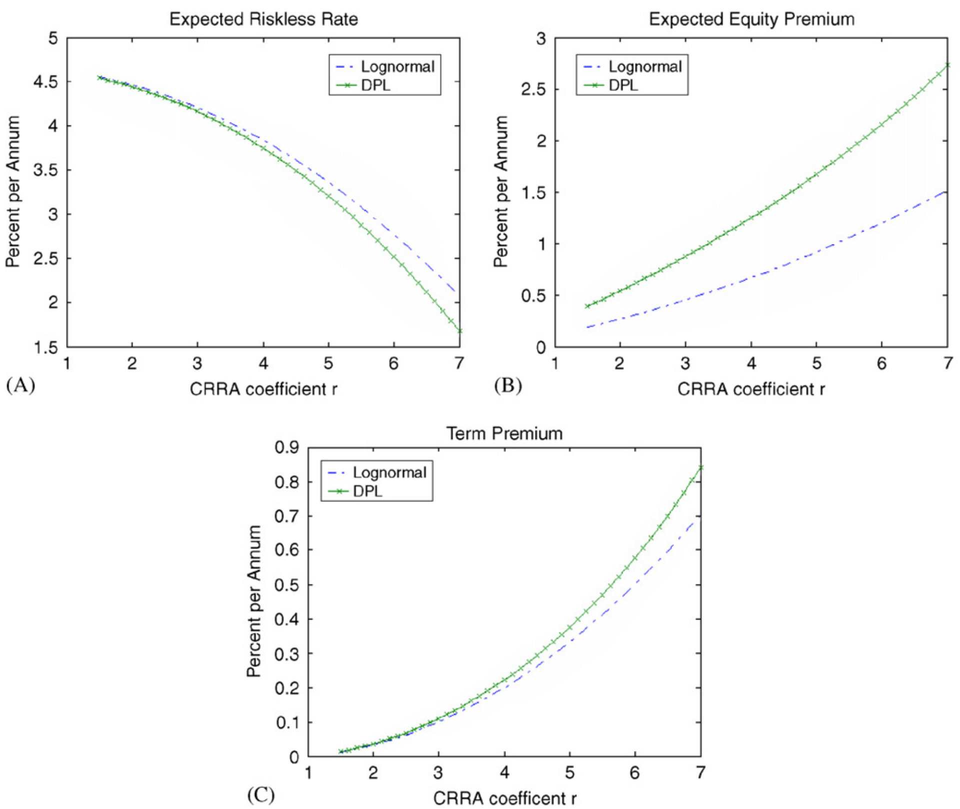

Fig. 2. Expected rates of return and term premia: sensitivity to risk aversion.

Table 4
Volatility of rates of return

		$r=1.5$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$
Volatility of risk-free rate	DPL	0.271	0.541	1.081	1.617	2.146	2.668	3.179
$\sigma\{R_{t+1}(1,0)\}$	Log-normal	0.261	0.522	1.042	1.557	2.066	2.568	3.061
	% Difference	3.7	3.7	3.8	3.8	3.9	3.9	3.9
Volatility of equity premium	DPL	3.819	4.137	4.845	5.618	6.431	7.268	8.118
$\sigma\{R_{t+1}(\infty,1)-R_{t+1}(1,0)\}$	Log-normal	3.595	3.889	4.549	5.273	6.037	6.824	7.625
	% Difference	6.3	6.4	6.5	6.5	6.5	6.5	6.5

Notes: All statistics are expressed in percent per annum. % Difference is the difference in the relevant statistic between the DPL and the log-normal cases, relative to the value in the log-normal case.

Table 4 presents the model-implied volatility (standard deviations) of equilibrium risk-free rates and the equity premium for the unrestricted DPL consumption growth rate process and the log-normal process. Volatility is expressed in percent per annum. As is evident from the table, the asset-pricing model is able to generate a volatility of the risk-free rate of 0.27 percent per annum with relative risk aversion coefficient of 1.5 and a volatility of the equity premium of 3.82 percent per annum. While the volatility of the risk-free rate increases to as much as 3.18 percent per annum at a risk aversion coefficient of 7, the volatility of the equity premium increases to only about 8.12 percent per annum at this high value for the risk aversion coefficient. The empirically observed volatility of the risk-free rate and the equity premium are 8.81 and 18.60 percent per annum respectively for the United States over this sample period as reported in Campbell (2003). Thus, the asset pricing model with habit formation is unable to generate sufficient variation in implied returns to conform to the empirically observed volatility. This is the well-known excess volatility puzzle.

Comparing the model-implied volatility of rates of return in the DPL and log-normal cases in Table 4, we find that accounting for fat tails leads to a higher volatility of both the risk-free rate and the equity premium. More specifically, the DPL model is able to increase volatility of the risk-free rate by up to 3.90 percent and that of the equity premium by up to 6.5 percent per annum compared to the log-normal case. Thus, accounting for fat tails moves the implied volatility of the equity premium towards a closer match with volatility of observed equity premium but not by enough.

Table 5 investigates sensitivity of the model-implied equilibrium expected risk-free rates, equity premium, and the term premium to changes in the value of the habit parameter h_1. The value of the risk aversion coefficient is fixed at a value of 4 in this exercise. In general, an increase in h_1 leads to a monotonic increase in all three implied rates of return for both the unrestricted DPL and the log-normal consumption growth rate processes. While the increase is modest for the mean risk-free rate (less than 0.1 percent per annum) as h_1 changes from 0.01 to 0.50, the mean equity premium increases by as much as 0.73 percent per annum and the term premium by 0.72 percent per annum. As the table makes clear, the impact of fat tails declines monotonically with increasing values of the habit parameter as the

Table 5
Expected rates of return and term premia

		$h_1 = 0.01$	$h_1 = 0.08$	$h_1 = 0.16$	$h_1 = 0.24$	$h_1 = 0.32$	$h_1 = 0.41$	$h_1 = 0.50$
Mean risk-free rate	DPL	3.751	3.747	3.750	3.760	3.776	3.804	3.841
$E\{R_{t+1}(1,0)\}$	Log-normal	3.841	3.838	3.839	3.848	3.863	3.888	3.921
	% Difference	−2.4	−2.4	−2.3	−2.3	−2.2	−2.15	−2.0
Mean equity premium	DPL	1.037	1.142	1.262	1.381	1.500	1.634	1.767
$E\{R_{t+1}(\infty,1)\}-E\{R_{t+1}(1,0)\}$	Log-normal	0.476	0.571	0.681	0.790	0.900	1.023	1.147
	% Difference	118.0	99.9	85.3	74.7	66.7	59.7	54.1
Term premium	DPL	0.015	0.119	0.237	0.355	0.472	0.603	0.735
$TP(n,0)$	Log-normal	0.013	0.107	0.213	0.320	0.427	0.548	0.668
	% Difference	11.6	11.3	11.0	10.7	10.5	10.2	9.9

Notes: All statistics are expressed in percent per annum. % Difference is the difference in the relevant statistic between the DPL and the log-normal cases, relative to the value in the log-normal case.

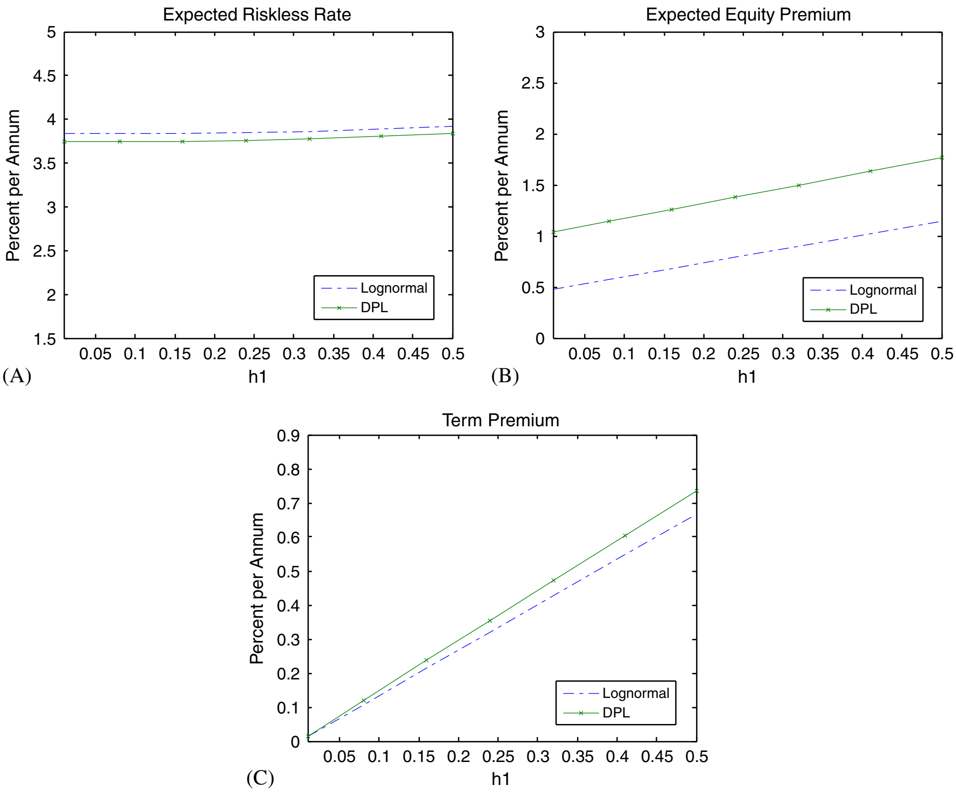

Fig. 3. Expected rates of return and term premia: sensitivity to habit.

Table 6
Volatility of rates of return

		$h_1 = 0.01$	$h_1 = 0.08$	$h_1 = 0.16$	$h_1 = 0.24$	$h_1 = 0.32$	$h_1 = 0.41$	$h_1 = 0.50$
Volatility of risk-free rate	DPL	0.107	0.861	1.725	2.591	3.463	4.450	5.445
$\sigma\{R_{t+1}(1{,}0)\}$	Log-normal	0.104	0.830	1.661	2.491	3.323	4.259	5.198
	% Difference	3.6	3.7	3.9	4.0	4.2	4.5	4.8
Volatility of Equity Premium	DPL	3.644	4.551	5.779	7.115	8.514	10.138	11.801
$\sigma\{R_{t+1}(\infty{,}1)-R_{t+1}(1{,}0)\}$	Log-normal	3.432	4.274	5.424	6.681	8.000	9.531	11.100
	% Difference	6.162	6.473	6.542	6.500	6.433	6.365	6.322

Notes: All statistics are expressed in percent per annum. % Difference is the difference in the relevant statistic between the DPL and the log-normal cases, relative to the value in the log-normal case.

differences between the rates of return with unrestricted DPL consumption growth rate process and the log-normal process shrink.

Fig. 3 plots the model-implied equilibrium mean risk-free rate, the mean equity and term premiums as a function of the habit parameter h_1 in both the DPL and the log-normal cases.

Table 6 investigates sensitivity of the model-implied volatility of equilibrium risk-free rates and the equity premium to changes in the value of the habit parameter h_1 for both the unrestricted DPL and the log-normal consumption growth rate processes. Volatility of both rates of return increases monotonically with h_1. With the DPL process, volatility of the risk-free rate increases from a low of 0.11 percent per annum to a high of 5.45 percent per annum as h_1 increases from 0.01 to 0.50. Volatility of the risk-free rate with the log-normal process is only slightly smaller in every instance. With the DPL process, volatility of the equity premium increases from a low of 3.64 percent/annum to a high of 11.80 percent/annum as $h_1 = 0.15$ increases from 0.01 to 0.50. Volatility of the risk-free rate with the log-normal process is once again a bit lower. Overall, although accounting for fat tails increases volatility of both rates of return, and this increase is greater at higher values of h_1, the increase is at most only 4.8 percent in the risk-free rates and 6.3 percent in the equity premium at a high value of $h_1 = 0.50$.

5. Conclusions

In this study we addressed the question: what are the costs of ignoring fat tails in the empirical distributions of macroeconomic time series on the equilibrium implications of macroeconomic models? We addressed this question within the context of the consumption-based asset-pricing model, modified to incorporate habit formation as in Abel (1999). We considered two versions of the model – one in which exogenous consumption evolves as a stochastic DPL process as in Wu (2006) and the other benchmark version in which consumption follows a Gaussian process. DPL nests α-stable distributions but has the advantage that all moments are finite under strictly positive dampening. This renders model-implied rates of return and equity and term premia finite under certain restrictions.

We parameterized the two versions of the model with estimates derived from the annual US monthly real per capita consumption data. Choosing suitable values for the preference parameters of the model, our results show that accounting for fat tails improves the ability of the asset-pricing model to replicate empirically observed mean risk-free rate, equity and the term premia. Specifically, accounting for fat tails through a DPL process generates 20 percent lower mean risk-free rate, 80 percent higher equity premium, and 20 percent higher term premium compared to the log-normal case. Accounting for fat tails also leads to an improvement in the model's ability to replicate empirically observed volatility of the risk-free and equity returns. While fat tails lead to an increase in the model-implied volatilities of these rates of return, these increases are more modest, at most 6.5 percent per annum.

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