L11

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Spin States and Generators

Spin $\frac{1}{2}$ $\quad \hat{R}\,|\psi_s\rangle = |\psi'_s\rangle$

$\hat{J}_i = \hat{S}_i$

$|s, m_s\rangle = |\tfrac{1}{2}\, m_s\rangle$

① $[\hat{S}_i, \hat{S}_j] = i \sum_k \varepsilon_{ijk} \hat{S}_k$

② $S^2 = S_x^2 + S_y^2 + S_z^2$; $[\hat{S}^2, \hat{S}_i] = 0$ $\quad [\hat{S}_z, S^2] = 0$

③ $\hat{S}_z |s, m_s\rangle = m_s |s, m_s\rangle$

④ $\hat{S}^2 |s, m_s\rangle = s(s+1)|s, m_s\rangle = \frac{1}{2}(\frac{1}{2}+1)|s, m_s\rangle = \frac{3}{4}|s, m_s\rangle$

⑤ $\hat{S}_\pm = \hat{S}_x \pm i\hat{S}_y$; $\hat{S}_\pm |s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}\,|s, m_{s\pm1}\rangle$

$\hat{S}_\pm |s, m_s\rangle = \sqrt{\frac{3}{4} - m_s(m_s \pm 1)}\,|s, m_{s\pm1}\rangle$

$\boxed{R(n,\theta)} = \boxed{e^{i \hat{S}_i n_i \theta}} = \boxed{D(n,\theta)}$

$\hat{S}_i \quad |s, m_s\rangle$

$|s, m_s\rangle \to |s, \tfrac{1}{2}\rangle$, $\to |s, -\tfrac{1}{2}\rangle$

take ③

$\langle s, m'| \hat{S}_z |s, m_s\rangle = m_s \langle s, m'| s, m_s\rangle$

$\langle s, m'| \hat{S}_z |s, m_s\rangle = m_s \langle s\, m'| s\, m\rangle = m\, \delta_{mm'}$

$\langle s\, m' | s\, m\rangle = \delta_{mm'}$

$$\boxed{\langle s, m'| \hat{S}_z |s, m_s\rangle = m\, \delta_{mm'}}$$

$\langle \psi_m | \hat{A} | \psi_n \rangle = A_{mn}$ → Matrix Element

$m = \frac{1}{2}, -\frac{1}{2}$

$\frac{1}{2} \to 1$

$-\frac{1}{2} \to 2$

$m' = \frac{1}{2}$, $m = \frac{1}{2}$

$\boxed{\langle s, \tfrac{1}{2}| S_z | s, \tfrac{1}{2}\rangle = \tfrac{1}{2}} = (S_z)_{\frac{1}{2}\frac{1}{2}} = (S_z)_{11} = \frac{1}{2}$

$m' = \frac{1}{2}$, $m = -\frac{1}{2}$

$\boxed{\langle s, \tfrac{1}{2}| S_z | s, -\tfrac{1}{2}\rangle} = -\frac{1}{2}\delta_{\frac{1}{2}-\frac{1}{2}} = \boxed{0} = (S_z)_{12} = 0$

$m' = -\frac{1}{2}$, $m = -\frac{1}{2}$

$\boxed{\langle s, -\tfrac{1}{2}| S_z | s -\tfrac{1}{2}\rangle} = -\frac{1}{2}\delta_{-\frac{1}{2}-\frac{1}{2}} = \boxed{-\tfrac{1}{2}} = (S_z)_{22} = -\frac{1}{2}$

$m' = -\frac{1}{2}$, $m = \frac{1}{2}$

$\boxed{\langle s, -\tfrac{1}{2}| S_z | s\, \tfrac{1}{2}\rangle = 0} = (S_z)_{21} = 0$

Generator of Rotation

$$\boxed{\hat{S}_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}}$$

$\hat{S}_x = ?$

$\hat{S}_y$

$\hat{S}_\pm |s, m\rangle = \sqrt{\frac{3}{4} - m(m \pm 1)}\;|s, m \pm 1\rangle$

$m = \frac{1}{2}$

$\hat{S}_+ |s, \tfrac{1}{2}\rangle = 0$ $\qquad (\hat{S}_+)_{11} = 0 \qquad (\hat{S}_+)_{21} = 0$

$\langle s, \tfrac{1}{2}| \hat{S}_+ |s, \tfrac{1}{2}\rangle = 0$; $\quad \langle s -\tfrac{1}{2}| \hat{S}_+ |s\, \tfrac{1}{2}\rangle = 0$

$m = -\frac{1}{2}$

$\hat{S}_+ |s, -\tfrac{1}{2}\rangle = \sqrt{\frac{3}{4} + \frac{1}{2}(-\frac{1}{2}+1)}\;|s, \tfrac{1}{2}\rangle = \sqrt{\frac{3}{4} + \frac{1}{4}}\;|s, \tfrac{1}{2}\rangle = |s, \tfrac{1}{2}\rangle \Rightarrow 0$ $\qquad (\hat{S}_+)_{22} = 0$

$\langle s, \tfrac{1}{2}| \hat{S}_+ |s -\tfrac{1}{2}\rangle = \langle s\, \tfrac{1}{2}| s\, \tfrac{1}{2}\rangle = 1$ $\qquad (\hat{S}_+)_{12} = 1$

$\hat{S}_+ = \begin{pmatrix} 0 & \\ 0 & \end{pmatrix}$

$m = \frac{1}{2}$

$\langle s \tfrac{1}{2}| \hat{S}_- |s, \tfrac{1}{2}\rangle = \sqrt{\frac{3}{4} - \frac{1}{2}(\frac{1}{2}-1)}\;|s, -\tfrac{1}{2}\rangle = |s, -\tfrac{1}{2}\rangle = 0$ $\qquad (\hat{S}_-)_{11} = 0$

$\langle s -\tfrac{1}{2}| s -\tfrac{1}{2}\rangle = 1$ $\qquad (\hat{S}_-)_{21} = 1$

$\hat{S}_- = ($

$m = -\frac{1}{2}$

$\hat{S}_- |s -\tfrac{1}{2}\rangle = \sqrt{\frac{3}{4} + \frac{1}{2}(-\frac{1}{2}-1)}\;|s -\tfrac{3}{2}\rangle = \boxed{0}$ $\qquad (\hat{S}_-)_{12} = 0 \qquad (\hat{S}_-)_{22} = 0$

$$\hat{S}_z = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \boxed{\hat{S}_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{S}_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}$$

$\hat{S}_x = \frac{1}{2}(\hat{S}_+ + \hat{S}_-)$

$$\hat{S}_+ = \hat{S}_x + i\hat{S}_y \qquad S_+ + S_- = 2S_x, \quad S_x = \frac{1}{2}(S_+ + S_-)$$
$$\hat{S}_- = \hat{S}_x - i\hat{S}_y \qquad \hat{S}_+ - \hat{S}_- = 2i\hat{S}_y, \quad \hat{S}_y = \frac{-i}{2}(\hat{S}_+ - \hat{S}_-)$$

$$\hat{S}_x = \frac{1}{2}\left[\overset{S_+}{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + \overset{S_-}{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}\right] = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_y = -\frac{i}{2}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$\hat{L} =$

$$\boxed{\hat{S}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \hat{S}_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

Pauli Matrices

$$\vec{\sigma}_i, \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\boxed{\hat{S}_i = \frac{1}{2}\sigma_i} \quad !$$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$\det M = ad - bc$$

1 $$[\hat{S}_i \, \hat{S}_j] = i\sum_k \varepsilon_{ijk} S_k$$

$$\frac{1}{4}[\sigma_i \, \sigma_j] = i\frac{1}{2}\sum_k \varepsilon_{ijk}\sigma_k \qquad \boxed{[\sigma_i \, \sigma_j] = 2i\sum_{ijk} \sigma_k} \leftarrow 1$$

$$\boxed{\sigma_i = \sigma_i^\dagger = \sigma_i^{-1}} \quad 2 \quad \boxed{\det \sigma_i = -1, \quad Tr\,\sigma_i = 0}$$

$$\boxed{\sigma_i^2 = I}$$

$$\boxed{(\hat{n}_i\sigma_i)^2 = (n_x\sigma_x + n_y\sigma_y + n_z\sigma_z)^2 = \boxed{I}}$$

$$n_x^2 + n_y^2 + n_z^2 = 1$$

$$\boxed{\sigma_i\sigma_j = \delta_{ij} + i\sum \varepsilon_{ijk}\sigma_k}$$

$$\boxed{\{\sigma_i\sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}}$$

Summarize.

$$\boxed{\hat{S}_i = \frac{1}{2}\sigma_i}$$

$|S, m_s\rangle = ?$

$$\boxed{\hat{S}_z |S, m_s\rangle = m_s |S, m_s\rangle}$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} |S, m_s\rangle = m_s |S, m_s\rangle$$

$\boxed{2\times 2} \cdot [2\times 1]$ Number (2×1) ~~Number~~ Vector

$|S, m_s\rangle = \boxed{2\times 2}$

$[2\times 1]$

$$|S, \tfrac{1}{2}\rangle = \begin{pmatrix} a \\ b \end{pmatrix}; \quad |S, -\tfrac{1}{2}\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$$

$\langle S, \frac{1}{2} | S, \frac{1}{2} \rangle = 1 \qquad (a \ b)\begin{pmatrix} a \\ b \end{pmatrix} = 1 = a^2 + b^2 = 1$

$\langle S, -\frac{1}{2} | S, -\frac{1}{2} \rangle = 1 \qquad (c \ d)\begin{pmatrix} c \\ d \end{pmatrix} = c^2 + d^2 = 1$

$\langle S, \frac{1}{2} | S, -\frac{1}{2} \rangle = 0 \qquad (a \ b)\begin{pmatrix} c \\ d \end{pmatrix} = ac + bd = 0$

$$\hat{S}_z |S, m_s\rangle = m_s |S, m_s\rangle$$

$m_s = \frac{1}{2}$
$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$m_s = -\frac{1}{2}$
$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} c \\ d \end{pmatrix}$$

$a = 1 \qquad b = 0$

$c = 0 \qquad d = 1$

$$\boxed{|S, \tfrac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |S, -\tfrac{1}{2}\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}} \ !!$$

$$\hat{S}^2|S, m_s\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)|S m_s\rangle \qquad \hat{S}_z|S, m_s\rangle = m_s|S, m_s\rangle$$

$$\hat{S}^2 = \frac{1}{2}(\hat{S}_+\hat{S}_- + \hat{S}_-\hat{S}_+) + \hat{S}_z^2$$

$$\hat{S}_x|\Psi^x\rangle = [\ ]|\Psi^x\rangle$$

$$\hat{S}_y|\Psi^y\rangle = [\ ]|\Psi^y\rangle$$

Sumarizing:

$$\hat{S}_i = \frac{1}{2}\sigma_i$$

$$\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \sigma_y = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}, \quad \sigma_z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$D^{(\frac{1}{2})}(n,\theta) = e^{-i\hat{S}_i n_i\theta} = e^{-i\frac{1}{2}\hat{\sigma}_i n_i\theta} = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-i\frac{1}{2}\hat{\sigma}_i\hat{n}_i\theta\right)^n =$$

$$= \underbrace{\sum_{\substack{n=0\\ \text{even}}}^{\infty}\frac{1}{n!}\left(-i\frac{1}{2}\hat{\sigma}_i\hat{n}_i\theta\right)^n}_{a} + \underbrace{\sum_{\substack{n=1\\ n-\text{odd}}}^{\infty}\frac{1}{n!}\left(-\frac{i}{2}\hat{\sigma}_i n_i\right)}_{b}$$

(a) $$1 + \frac{\left(-i\frac{1}{2}\hat{\sigma}_i\hat{n}_i\theta\right)^2}{2} + \frac{1}{4!}\left(-i\frac{1}{2}\hat{\sigma}_i\hat{n}_i\theta\right)^4 + \cdots = 1 - \frac{\left(\frac{\theta}{2}\right)^2}{2} + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 - \frac{1}{6}$$

$$-\frac{1}{4}\frac{(\sigma_i n_i)^2\theta^2}{2} = -\frac{1}{4}\frac{\theta^2}{2} = -\frac{\left(\frac{\theta}{2}\right)^2}{2}$$

$$\frac{1}{4!}\left(-i\frac{1}{2}\sigma_i n_i\theta\right)^2\left(-i\frac{1}{2}\sigma_i n_i\theta\right)^2 = \frac{1}{4!}\left(\frac{\theta}{2}\right)^4$$

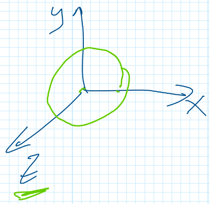
(b) $$-i\frac{\sigma_i n_i}{2}\theta + \frac{1}{3!}\left(-i\frac{\sigma_i n_i\theta}{2}\right)^3 + \frac{1}{5!}\left(\frac{-i\sigma_i n_i\theta}{2}\right)^5$$

$$= -i\sigma_i n_i\left(I\frac{\theta}{2} + \frac{1}{3!}\underbrace{(-i\sigma_i n_i)^2}_{-(\sigma_i n_i)^2 = I}\left(\frac{\theta}{2}\right)^3 + \frac{1}{5!}\underbrace{(-i\sigma_i n_i)^4}_{+((\sigma_i n_i)^2)^2 = I}\left(\frac{\theta}{2}\right)^5\right)$$

$$= -i\sigma_i n_i\left(I\frac{\theta}{2} - \frac{1}{3!}\left(\frac{\theta}{3}\right)^3 + \frac{1}{5!}\left(\frac{\theta}{2}\right)^5\right) = -i\sigma_i n_i\ S$$

$$\ldots\left(\frac{\theta}{\ }\right) \ldots (\hat{\sigma}\cdot n)\sin\frac{\theta}{\ }$$

$$\boxed{D(n,\theta) = I\cos\left(\frac{\theta}{2}\right) - i(\sigma \cdot n)\sin\frac{\theta}{2}}$$



$$n_1 = n_z$$

$$D(n_z, 2\pi) = I\cos(\pi) - i\sigma_z \underline{\sin\pi}$$

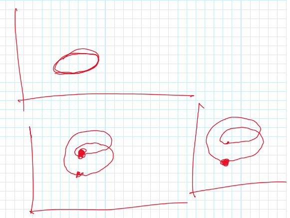
$$\cos(\pi) = -1, \quad \sin\pi = 0$$

$$4\pi$$

$$D(n_z,\theta) = I\cos\left(\frac{\theta}{2}\right) - i\sigma_z\sin\frac{\theta}{2}$$

$$D(n_z,2\pi)|s,m_s\rangle = -|s,m_s\rangle$$

$$\boxed{D(n_z,\theta)|s,m_s\rangle = |s',m_s\rangle}$$



$$R(n_z,\theta) = e^{-i\hat{L}n_z\theta} = e^{-iL_z\theta}$$

$$\boxed{R(n_z,\theta)|\psi_{em}(\theta,\varphi)\rangle = e^{-i\hat{L}_z\theta}|\psi_{em}(\theta,\varphi)\rangle} = e^{-im\ldots}$$

$$e^{-i\hat{L}_z\theta}|\psi_{em}(\theta,\varphi)\rangle = \sum\frac{(-i\hat{L}_z\theta)^n}{n!}|\psi_{em}(\theta)\rangle =$$

$$\hat{L}_z|\psi_{em}(\theta,\varphi)\rangle = m|\psi_{em}(\theta,\varphi)\rangle$$

$$= \sum\frac{(-im\theta)^n}{n!}|\psi_{em}(\theta,\varphi)\rangle$$

$$e^{-im\theta}|\psi_{em}(\theta,\varphi)\rangle$$