Total Angular Momentum

L13

Wednesday, March 3, 2021 5:00 PM

$J = L + S$

$|\psi_{\ell m_\ell}\rangle \otimes |s, m_s\rangle$

Example 1: $s = \frac{1}{2}$, $\ell = 0$

$|\phi_{J m_J}\rangle = |\psi_{\ell m_\ell, s m_s}\rangle$ $\quad |\psi_{\ell m_\ell s m_s}\rangle$

$J^2 |\phi_{J, m_J}\rangle = j(j+1)|\phi_{J, m_J}\rangle$

$J_z |\phi_{J, m_J}\rangle = m_J |\phi_{J, m_J}\rangle$

$\boxed{|\phi_{J m_J}\rangle = |\psi_{00, s, m_s}\rangle} = |\psi_{00, s, m_s}\rangle$

$\boxed{\begin{matrix} J = s \\ m_J = m_s \end{matrix}}$

$|\phi_{J, m_J}, (\ell\, s)\rangle$

$(L_z + S_z)|\psi_{00 s m_s}\rangle = \underbrace{L_z |\psi_{00 s m}\rangle}_{0} + S_z |\psi_{00 s m}\rangle = m_s |\psi_{00 s m_s}\rangle$

$m_J |\phi_{J, m_J}\rangle$

$\hat{J}_z |\psi_{00 s m}\rangle = m_J |\psi_{00 s m_s}\rangle = m_s |\psi_{00 s m_s}\rangle$

$$\boxed{m_J = m_s}$$

$L_+ = L_x + iL_y \quad | \quad S_+ = S_x + iS_y$

$L_- = L_x - iL_y \quad | \quad S_- = S_x - iS_y$

$LS = \frac{L_+ S_-}{2} + \frac{L_- S_+}{2} + L_z S_z$

$J^2 |\phi_{J m_J}\rangle = J^2 |\psi_{00 s m_s}\rangle$

$J^2 = (L+S)^2 = L^2 + L \cdot S + S \cdot L + S^2 = \hat{L}^2 + 2\hat{L}\hat{S} + \hat{S}^2 = //$

$J^2 = // = L^2 + \hat{L}_+ \hat{S}_- + \hat{L}_- \hat{S}_+ + 2L_z S_z + S^2$

$J^2 |\psi_{00 s m_s}\rangle = (L^2 + L_+ S_- + L_- S_+ + 2L_z S_z + S^2)|\psi_{00 s m_s}\rangle =$

$= \underbrace{L^2 |\psi_{00 s m_s}\rangle}_{\ell(\ell+1)|\psi_{00 m_s}\rangle = 0} + \underbrace{L_+ S_- |\psi_{00 s m_s}\rangle}_{0} + \underbrace{L_- S_+ |\psi_{00 s m_s}\rangle}_{0}$

$+ \underbrace{2L_z S_z |\psi_{00 s m_s}\rangle}_{0} + S^2 |\psi_{00 s m_s}\rangle = s(s+1)|\psi_{00 s m_s}\rangle$

$\| \; j(j+1)|\psi_{00 s m_s}\rangle$

$J = S$

$$\boxed{\hat{J}_\pm |\phi_{J, m_J}\rangle = \sqrt{j(j+1) - m(m \pm 1)}\, |\phi_{J, m_J \pm 1}\rangle}$$

$$\boxed{J = S}$$

$L_+ |\psi_{00 s m_s}\rangle = \sqrt{\ell(\ell+1) - m_\ell(m_\ell \pm 1)}\, |\psi_{\ell \pm 1, s m}\rangle$

$\sqrt{0(0+1) - 0(0\pm 1)} = 0$

$L_z |\psi_{00 m_s}\rangle = \underbrace{m_\ell}_{0} |\psi_{00 m_s}\rangle$

$S^2 |\psi_{00 s, m_s}\rangle = s(s+1)|\psi_{00 s m_s}\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)|\psi_{00 s m_s}\rangle$, $s = \frac{1}{2}$

$\frac{3}{4}|\psi_{00, s m_s}\rangle$

Summarizing

$$\boxed{s = \frac{1}{2}, \quad \ell = 0}$$

$$\boxed{\begin{matrix} |\phi_{J m_J}\rangle = |\psi_{00, s, m_s}\rangle \\ J = S, \quad m_J = m_s \end{matrix}}$$

$|\psi_{\ell m_\ell; s m_s}\rangle$

Example 2

$J = L + S$

$s = \frac{1}{2}$, $\ell = 1$ — we know

$|\phi_{J m_J}\rangle = ?$

$|\psi_{\ell m_\ell, s m_s}\rangle = |\psi_{\ell m_\ell}\rangle \otimes |s, m_s\rangle$ — we know

$L^2 |\psi_{\ell m_\ell, s m_s}\rangle = \overset{2}{\ell(\ell+1)} |\psi_{\ell m_\ell s m_s}\rangle$

a) $J^2 |\phi_{J m_J}\rangle = j(j+1)|\phi_{J m_J}\rangle$

b) $J_z |\phi_{J m_J}\rangle = m_J |\phi_{J m_J}\rangle$

$L_z |\psi_{\ell, m_\ell, s m_s}\rangle = \overset{-1,0,1}{m_\ell} |\psi_{\ell, m_\ell s m_s}\rangle$

$S^2 |\psi_{\ell, m_\ell, s m_s}\rangle = \overset{\frac{1}{2}(\frac{1}{2}+1) = 3/4}{s(s+1)} |\psi_{\ell, m_\ell, s m_s}\rangle$

$S_z |\psi_{\ell, m_\ell, s m_s}\rangle = \overset{-\frac{1}{2}, \frac{1}{2}}{m_s} |\psi_{\ell, m_\ell, s m_s}\rangle$

b) $J_z |\phi_{J m_J}\rangle = (L_z + S_z)|\phi_{J m_J}\rangle$

Assume $|\phi_{J m_J}\rangle = |\psi_{\ell m_\ell, s, m_s}\rangle$

$J_z |\phi_{J m_J}\rangle = (L_z + S_z)|\psi_{\ell m_\ell s m_s}\rangle = (m_\ell + m_s)|\psi_{\ell m_\ell s m_s}\rangle$

$m_j |\psi_{e, m_e, s m_s}\rangle = (m_e + m_s)|\psi_{e, m_e, s m_s}\rangle$

$\begin{cases} L_\pm |\psi_{e m_e, s m_s}\rangle = \sqrt{e(e+1) - m_e(m_e \pm 1)}\, |\psi_{e m_e \pm 1; s m_s}\rangle \\ S_\pm |\psi_{e m_e, s m_s}\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)}\, |\psi_{e m_e, s m_s \pm 1}\rangle \end{cases}$

$\boxed{m_j = m_e + m_s}$

$\Phi_{J m_j} \neq |\psi_{e m_e, s m_s}\rangle$

@ $J^2 |\psi_{e m_e s m_s}\rangle = (\underset{I}{L^2} + \underset{II}{L_+ S_-} + \underset{III}{L_- S_+} + \underset{IV}{2 L_z S_z} + \underset{V}{S^2}) |\psi_{e m_e s m_s}\rangle =$

$|\psi_{1 1; \frac{1}{2}\frac{1}{2}}\rangle = \overset{I}{e(e+1)} |\psi_{e m_e s m_s}\rangle + \overset{II}{\sqrt{e(e+1) - m_e(m_e+1)}\sqrt{s(s+1) - m_s(m_s-1)}}\, |\psi_{e (m_e+1) s m_s - 1}\rangle$

$1(1+1) |\psi_{1 1; \frac{1}{2}\frac{1}{2}}\rangle + \sqrt{1(1+1) - 1(1+1)}\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\, |\psi_{1 2; \frac{1}{2} -\frac{1}{2}}\rangle \to 0$

III $\sqrt{e(e+1) - m_e(m_e-1)}\sqrt{s(s+1) - m_s(m_s+1)}\, |\psi_{e m_e - 1\, s m_s + 1}\rangle + 2 m_e m_s |\psi_{e m_e s m_s}\rangle$

$\sqrt{1(1+1) - 1(1-1)}\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}+1)}\, |\psi_{1 0; \frac{1}{2} \frac{3}{2}}\rangle \to 0$ | $2 \cdot 1 \cdot \frac{1}{2} |\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle$

$+ \overset{V}{s(s+1)} |\psi_{e m_e s m_s}\rangle$

$\frac{1}{2}(\frac{1}{2}+1)$

$s = \frac{1}{2}$ $\quad e = 1$

$\Rightarrow$ Top State Theorem

Take Top state of $|\psi_{e m_e, s m_s}\rangle$

$|\psi_{1 m_e, \frac{1}{2} m_s}\rangle$ (max, max)

Top $\quad \boxed{|\psi_{1 1; \frac{1}{2}\frac{1}{2}}\rangle}$

$J^2 |\psi_{1 1, \frac{1}{2}\frac{1}{2}}\rangle = 2 |\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle + 1 |\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle + \frac{3}{4} |\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle =$

Top

$= (2 + 1 + \frac{3}{4}) |\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle$

$j(j+1) |\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle \qquad \boxed{\frac{3}{2}(\frac{3}{2}+1)} \boxed{|\psi_{1 1 \frac{1}{2}\frac{1}{2}}\rangle}$

$1 + \frac{1}{2}$

$\boxed{j = \frac{3}{2}} = \boxed{1 + \frac{1}{2}}$

$e = 1$
$s = \frac{1}{2}$
$j = \frac{3}{2}$

$\Rightarrow \boxed{|\Phi_{\frac{3}{2}\frac{3}{2}}\rangle = |\Phi_{J m_j}\rangle = |\psi_{1 1 \frac{1}{2} \frac{1}{2}}\rangle}$ (m_e, m_s) Top State

$J = 1 + \frac{1}{2} = \frac{3}{2}$

$m_j = \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$

$|\Phi_{J m_j}\rangle \Rightarrow \boxed{J = \frac{3}{2}}$ $\quad m_j$: $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, $-\frac{3}{2}$

$|\Phi^{j\ m_s}_{\frac{3}{2}\frac{3}{2}}\rangle$ — state ✓ Top state

$\rightarrow |\phi_{\frac{3}{2}\frac{1}{2}}\rangle$ — ✓ 4 — states

$|\phi_{\frac{3}{2}-\frac{1}{2}}\rangle$ ✓

$|\phi_{\frac{3}{2}-\frac{3}{2}}\rangle$ — $J_-|\phi_{\frac{3}{2}-\frac{1}{2}}\rangle$ — HW

$\Rightarrow$ Using step down operation

$$J_-|\phi_{\frac{3}{2}\frac{3}{2}}\rangle = \sqrt{j(j+1)-m_j(m_j-1)}\,|\phi_{\frac{3}{2}\,\frac{3}{2}-1}\rangle$$

$m_j - 1$

$$\sqrt{\tfrac{3}{2}(\tfrac{3}{2}+1)-\tfrac{3}{2}(\tfrac{3}{2}-1)}\,|\phi_{\frac{3}{2}\frac{1}{2}}\rangle$$

$$\tfrac{3}{2}(\cancel{\tfrac{3}{2}}+1-\cancel{\tfrac{3}{2}}+1)=3$$

$$\boxed{J_-|\phi_{\frac{3}{2}\frac{3}{2}}\rangle} = \boxed{\sqrt{3}\,|\phi_{\frac{3}{2}\frac{1}{2}}\rangle}$$

$\ell \; m_\ell \; s \; m_s$

$$(\hat{L}_- + \hat{S}_-)|\psi_{11,\frac{1}{2}\frac{1}{2}}\rangle = L_-|\psi_{11\frac{1}{2}\frac{1}{2}}\rangle + S_-|\psi_{11\frac{1}{2}\frac{1}{2}}\rangle = ''$$

$$'' \sqrt{\ell(\ell+1)-m_\ell(m_\ell-1)}\,|\psi_{10\frac{1}{2}\frac{1}{2}}\rangle + \sqrt{s(s+\tfrac{1}{2})-m_s(m_s-1)}\,|\psi_{11\,\frac{1}{2}-\frac{1}{2}}\rangle$$

$$\sqrt{1(1+1)-\cancel{1(1-1)}}\,|\psi_{10\frac{1}{2}\frac{1}{2}}\rangle + \sqrt{\tfrac{1}{2}(\tfrac{1}{2}+1)-\tfrac{1}{2}(\tfrac{1}{2}-1)}\,|\psi_{11;\frac{1}{2}-\frac{1}{2}}\rangle$$

$$\tfrac{3}{4}+\tfrac{1}{4}=1$$

$$= \boxed{\sqrt{2}\,|\psi_{10,\frac{1}{2}\frac{1}{2}}\rangle + |\psi_{11;\frac{1}{2}-\frac{1}{2}}\rangle}$$

$$\sqrt{3}\,|\phi_{\frac{3}{2}\frac{1}{2}}\rangle = \sqrt{2}\,|\psi_{10,\frac{1}{2}\frac{1}{2}}\rangle + |\psi_{11;\frac{1}{2}-\frac{1}{2}}\rangle$$

$$\boxed{|\phi_{\frac{3}{2}\frac{1}{2}}\rangle = \sqrt{\tfrac{2}{3}}\,|\psi_{10,\frac{1}{2}\frac{1}{2}}\rangle + \tfrac{1}{\sqrt{3}}\,|\psi_{11;\frac{1}{2}-\frac{1}{2}}\rangle}$$

$j \; m_j$ $\quad$ $\ell \; m_\ell \; s \; m_s$ $\quad$ $\ell \; m_\ell, \; s, \; m_s$

!

$$\boxed{m_j = m_\ell + m_s}$$

$$\boxed{J_-|\phi_{\frac{3}{2}\frac{1}{2}}\rangle} = \sqrt{j(j+1)-m_j(m_j-1)}\,|\phi_{\frac{3}{2}-\frac{1}{2}}\rangle = \boxed{2\,|\phi_{\frac{3}{2}-\frac{1}{2}}\rangle}$$

$$(L_- + S_-)\cdot\left(\sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}\frac{1}{2}}\right\rangle + \frac{1}{\sqrt{3}}\left|\psi_{11;\frac{1}{2}-\frac{1}{2}}\right\rangle\right) =$$

$$= L_-\left(\sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}\frac{1}{2}}\right\rangle + \frac{1}{\sqrt{3}}\left|\psi_{11;\frac{1}{2}-\frac{1}{2}}\right\rangle\right) +$$

$$+ S_-\left(\sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}\frac{1}{2}}\right\rangle + \frac{1}{\sqrt{3}}\left|\psi_{11;\frac{1}{2}-\frac{1}{2}}\right\rangle\right) =$$

$-3/2$

$$= \underbrace{\sqrt{l(l+1)-m_l(m_l-1)}}_{\sqrt{2}}\sqrt{\frac{2}{3}}\left|\psi_{1-1;\frac{1}{2}\frac{1}{2}}\right\rangle + \underbrace{\sqrt{l(l+1)-m_l(m_l-1)}}_{\sqrt{2}}\frac{1}{\sqrt{3}}\left|\psi_{1,0;\frac{1}{2}-\frac{1}{2}}\right\rangle$$

$$+ \underbrace{\sqrt{s(s+1)-m_s(m_s-1)}}_{1}\sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}-\frac{1}{2}}\right\rangle + \underbrace{\cancel{\sqrt{s(s+1)-m_s(m_s-1)}}}_{0}\frac{1}{\sqrt{3}}\left|\psi_{11;\frac{1}{2}-\frac{3}{2}}\right\rangle =$$

$$= \frac{2}{\sqrt{3}}\left|\psi_{1-1;\frac{1}{2}\frac{1}{2}}\right\rangle + 2\sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}-\frac{1}{2}}\right\rangle$$

$$2\left|\phi_{\frac{3}{2}-\frac{1}{2}}\right\rangle = \frac{2}{\sqrt{3}}\left|\psi_{1-1;\frac{1}{2}\frac{1}{2}}\right\rangle + 2\sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}-\frac{1}{2}}\right\rangle$$

$$\boxed{\left|\phi_{\frac{3}{2}-\frac{1}{2}}\right\rangle = \frac{1}{\sqrt{3}}\left|\psi_{1-1;\frac{1}{2}\frac{1}{2}}\right\rangle + \sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}-\frac{1}{2}}\right\rangle}$$

$$J_-\left|\phi_{\frac{3}{2}-\frac{1}{2}}\right\rangle = \underbrace{\sqrt{j(j+1)-m_j(m_j-1)}}_{\sqrt{3}}\left|\phi_{\frac{3}{2}-\frac{3}{2}}\right\rangle = \sqrt{3}\left|\phi_{\frac{3}{2}-\frac{3}{2}}\right\rangle$$

$$(L_- + S_-)\left(\frac{1}{\sqrt{3}}\left|\psi_{1-1;\frac{1}{2}\frac{1}{2}}\right\rangle + \sqrt{\frac{2}{3}}\left|\psi_{10;\frac{1}{2}-\frac{1}{2}}\right\rangle\right)$$

$$\boxed{|\Phi_{\frac{3}{2}\,-\frac{3}{2}}\rangle = |\Psi_{1\,-1;\,\frac{1}{2}\,-\frac{1}{2}}\rangle}$$ HW

$\Rightarrow$ Counting of states

$|\Psi_{11}\rangle$
$|\Psi_{10}\rangle$
$|\Psi_{1-1}\rangle$

$|s,\frac{1}{2}\rangle$ $|s,$

$|\Psi_{1\,m_\ell,\,s,m_s}\rangle = |\Psi_{1 m_\ell}\rangle \otimes |s, m_s\rangle$

6 - states

$|\phi_{\frac{3}{2}\frac{3}{2}}\rangle$ $\quad j=\frac{3}{2}\ m_j=\frac{3}{2}$

$|\phi_{\frac{3}{2}\frac{1}{2}}\rangle$

$|\phi_{\frac{3}{2}\,-\frac{1}{2}}\rangle$ $\quad \boxed{4\text{ states}}$ $\quad j=\frac{3}{2}\quad m_j=-\frac{1}{2}$

$|\phi_{\frac{3}{2}\,-\frac{3}{2}}\rangle$

$\boxed{m_j = m_\ell + m_s}$

$m_j = 3/2 \longrightarrow j=\frac{3}{2}$

$m_j = \frac{1}{2} \longrightarrow j=\frac{3}{2},\ j=\frac{1}{2}$

2 - missing ϕ states

Always $\boxed{m_j = m_\ell + m_s}$

Top state $m_j = 1+\frac{1}{2} = \frac{3}{2}$

$\downarrow$ max

$m_j^{max} = j \longrightarrow \boxed{j=\frac{3}{2}}$

$\Longrightarrow$ $\quad m_j = m_\ell + m_s$

one

$\boxed{j=\frac{1}{2}}$ $\quad m_j = \frac{1}{2}$

$\boxed{m_j = 1 - \frac{1}{2} = \frac{1}{2}}$

one

$j = \frac{1}{2}; \quad m_j = -\frac{1}{2}$

2

$m_j = -1 + \frac{1}{2} = -\frac{1}{2}$

$\Rightarrow \quad \boxed{J = L + S}$

$J = 1 + \frac{1}{2}$

$\boxed{\frac{3}{2}}$ $\boxed{\frac{1}{2}}$

4-states

2-states

$m_j = m_\ell + m_s$

$|\phi_{\frac{1}{2}\frac{1}{2}}\rangle = \alpha|\psi_{10;\frac{1}{2}\frac{1}{2}}\rangle + \beta|\psi_{11;\frac{1}{2}-\frac{1}{2}}\rangle$

$m_j = m_\ell + m_s$

$|\phi_{\frac{1}{2}-\frac{1}{2}}\rangle = \gamma|\psi_{10;\frac{1}{2}-\frac{1}{2}}\rangle + \delta|\psi_{1-1;\frac{1}{2}\frac{1}{2}}\rangle$

$\Rightarrow \quad \boxed{\alpha^2 + \beta^2 = 1} \quad \boxed{\gamma^2 + \delta^2 = 1}$

from above

$\langle\phi_{\frac{3}{2}\frac{1}{2}}|\phi_{\frac{1}{2}\frac{1}{2}}\rangle = 0 \Rightarrow$

$\langle\phi_{\frac{3}{2}-\frac{1}{2}}|\phi_{\frac{1}{2}-\frac{1}{2}}\rangle = 0$

j $\quad$ j $\quad$ $\phi_{\frac{1}{2}\frac{1}{2}}$

$$\left(\sqrt{\frac{2}{3}}\langle\psi_{10,\frac{1}{2}\frac{1}{2}}| + \frac{1}{\sqrt{3}}\langle\psi_{11,\frac{1}{2}-\frac{1}{2}}|\right)\left|\alpha|\psi_{10\frac{1}{2}\frac{1}{2}}\rangle + \beta|\psi_{11\frac{1}{2}}\right.$$

$$=\sqrt{\frac{2}{3}}\alpha + \frac{1}{\sqrt{3}}\beta = 0$$

$$\begin{cases}\sqrt{\frac{2}{3}}\alpha + \frac{1}{\sqrt{3}}\beta = 0 \\ \alpha^2+\beta^2 = 1\end{cases} \Longrightarrow \boxed{\alpha = -\frac{1}{\sqrt{2}}\beta} \Rightarrow$$

$$\frac{1}{2}\beta^2+\beta^2=1$$

$$\frac{3}{2}\beta^2 = 1$$

$$\boxed{\beta=\sqrt{\frac{2}{3}}}$$

$$\boxed{\alpha = -\frac{1}{\sqrt{3}} \qquad \beta = \sqrt{\frac{2}{3}}}$$

$$\boxed{J_- \left(|\phi_{\frac{1}{2}\frac{1}{2}}\rangle = \sqrt{\frac{1}{3}}\left(\sqrt{2}|\psi_{11\frac{1}{2}-\frac{1}{2}} - |\psi_{10\frac{1}{2}\frac{1}{2}}\rangle\right)\right)}$$

$(L_- + S_-)$

$$\boxed{|\phi_{\frac{1}{2}-\frac{1}{2}}\rangle = \frac{1}{\sqrt{3}}\left[|\psi_{10,\frac{1}{2}-\frac{1}{2}}\rangle - \sqrt{2}|\psi_{1-1,\frac{1}{2}\frac{1}{2}}\rangle\right.}$$