L24

April 17, 2021

⇒ Summarizing from the last lecture

$\langle n_i \rangle = \dfrac{1}{e^{\alpha+\beta\varepsilon_i} \mp 1}$ — Bosons (−), Fermions (+) $\qquad \beta = \frac{1}{KT}$

– Introduce $\alpha = -\mu\beta$ $\quad$ μ – chemical Potential

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(\varepsilon_i-\mu)} \mp 1}} \qquad \beta = \frac{1}{KT}$$

μ – chemical Potential

We consider now systems consisting of infinite number of particles

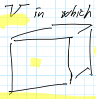
– For Probability of finding $\langle n_i \rangle$ particles in the state i we obtained

$$\boxed{\langle n_i \rangle = \frac{1}{e^{\beta(\varepsilon_i-\mu)} \mp 1}}$$

(− Bosons, + Fermions)

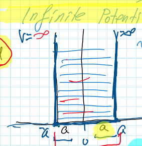
– How to describe infinite number of particles

① Consider volume V in which particles are confined



Quantum Mechanically it corresponds to 3-d Infinite Potential well

1d



$V=\infty \qquad V=\infty$

$\psi_n^+(x) = \cos\left(\frac{n\pi x}{2a}\right) = \cos(K_{nx} x)$ $\quad$ n-odd

$\psi_n^-(x) = \sin\left(\frac{n\pi x}{2a}\right) = \sin(K_{nx} x)$ $\quad$ n-even $\qquad n_x = 1, 2 \ldots$

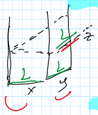
$$\boxed{K_{nx} = \frac{n\pi}{2a}}$$

$L = 2a$

$n = 1, 2 \ldots \quad K_{nx} = \frac{P_{nx}}{\hbar} \Rightarrow$

$\boxed{E_n = \frac{P_{nx}^2}{2m}}$ $\qquad \boxed{P_{nx}} = \hbar K_{nx} = \boxed{\frac{n\pi\hbar}{2a}} = \boxed{\frac{n\pi\hbar}{L}}$

3d



$V = L^3$

$\boxed{P_{nx} = \frac{n_x \pi\hbar}{L}}, \ \boxed{P_{ny} = \frac{n_y \pi\hbar}{L}}, \ \boxed{P_{nz} = \frac{n_z \pi\hbar}{L}}$

$n_i = 1, \ldots \infty)$ $\quad P_{nz} = \frac{n_z\pi\hbar}{L} \quad dP_{nz} = \frac{dn_z \pi\hbar}{L}$

$E_n = \frac{P_{nx}^2}{2m} + \frac{P_{ny}^2}{2m} + \frac{P_{nz}^2}{2m} =$

$= \frac{\pi^2\hbar^2}{2mL^2}(n_x^2 + n_y^2 + n_z^2)$

$\boxed{n^2 = n_x^2 + n_y^2 + n_z^2}$

$\boxed{E_n} = \frac{\pi^2\hbar^2}{2mL^2} n^2 = $

$V = L^3 \qquad = \boxed{\frac{\pi^2\hbar^2 n^2}{2m V^{2/3}}} \qquad n \sim ?$

Consider Helium Gas

$E_n \approx 5\times10^{-21} eV$

$V = 1$

$\boxed{n \approx 2\cdot10^{9}} =$

– Lets Consider $\sum\limits_{n} = \sum\limits_{n_x}^{N}\sum\limits_{n_y}^{N}\sum\limits_{n_z}^{N} \approx \int\limits_{1}^{\infty} dn_x \int\limits_{1}^{\infty} dn_y \int\limits_{1}^{\infty} dn_z = $

$dn_x = \frac{L}{\pi\hbar} \cdot dP_x \qquad$ if $n_x, n_y, n_z \gg 1$

$= L^3 \int\limits_0^\infty \frac{dP_x}{\pi\hbar} \int\limits_0^\infty \frac{dP_y}{\pi\hbar} \int\limits_0^\infty \frac{dP_z}{\pi\hbar} = L^3 \int\limits_{-\infty}^{+\infty} \frac{dP_x}{2\pi\hbar} \int\limits_{-\infty}^{+\infty} \frac{dP_y}{2\pi\hbar} \int\limits_{-\infty}^{+\infty} \frac{dP_z}{2\pi\hbar} = $

$= V \int \frac{d^3P}{(2\pi\hbar)^3}$ — Momentum Phase space $\qquad \boxed{\frac{d^3P}{(2\pi\hbar)^3}}$

– if one discusses an Infinite number of particles in finite volume one expects $n_x, n_y, n_z \gg 1$

Thus the sum over the quantum States is defined by $\boxed{V \int \frac{d^3P}{(2\pi\hbar)^3}}$!

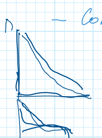
⇒ System with Infinite Number of Fermions

$$\langle n_i \rangle = \frac{1}{e^{\beta(\varepsilon_i-\mu)} + 1}, \qquad \beta = \frac{1}{KT}$$

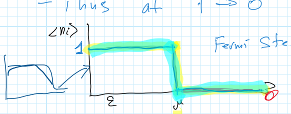
– Consider $T \to 0$, then $\beta \to \infty$

$\langle n_i \rangle \big|_{T\to 0} \to 0 \quad$ if $\boxed{\varepsilon_i > \mu}$ $\qquad e^{\beta|\varepsilon_i-\mu|} \to \infty$, $\ \beta \to \infty$

$\langle n_i \rangle \big|_{T\to 0} \to \boxed{1} \quad$ if $\boxed{\varepsilon_i < \mu}$ $\qquad e^{-\beta|\varepsilon_i-\mu|} \to 0$, $\ \beta \to \infty$



– Thus at $T \to 0$



Fermi Step Distribution

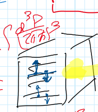
$$\boxed{\langle n_i \rangle = \theta(\mu - \varepsilon_i)} \qquad \boxed{\theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}} \ !$$

$\mu = \varepsilon_F$ – Fermi Energy

– $\boxed{E_F \equiv \varepsilon_F}$

– Introducing Fermi Momentum $P_F = \sqrt{2m E_F}$ $\qquad \boxed{E = \frac{P^2}{2m}}$

– how to calculate P_F ?



$V \int \frac{d^3P}{(2\pi\hbar)^3}$ $\quad \sum\limits_n$ $\quad E_F$

Consider Finite Volume V

Total number of Fermions (Electrons) will be

spin up + down

$$N = \sum_i \langle n_i\rangle \frac{d^3p}{(2\pi)^3} = 2V\int \theta(\varepsilon_F - \varepsilon)\frac{d^3p}{(2\pi)^3} = 2V\int \frac{d^3p}{(2\pi)^3} = //$$

$$// = 2V\int_0^{p_F} \frac{p^2\,dp\,d\Omega}{(2\pi)^3} = \frac{8\pi V}{(2\pi)^3}\int_0^{p_F} p^2\,dp = \frac{8\pi V}{(2\pi)^3}\frac{p_F^3}{3} = \boxed{\frac{V}{\pi^2}\frac{p_F^3}{3}}$$

$$\boxed{N = \frac{V}{3\pi^2}p_F^3} \Rightarrow \frac{N}{V} = \frac{p_F^3}{3\pi^2} \Rightarrow \boxed{p_F = (3\pi^2 \rho_N)^{1/3}} \qquad \boxed{E_F = \frac{p_F^2}{2m}}$$

ρ_N

$$E_F = \frac{1}{2m}(3\pi^2\rho_N)^{2/3}$$

— What it means $T \to 0$ in reality

$$\boxed{\langle n_i\rangle = \frac{1}{e^{\beta(\varepsilon_i - \mu)}+1}} = \frac{1}{e^{\frac{(\varepsilon_i-\mu)}{KT}}+1} =$$

$$= \frac{1}{e^{\frac{\varepsilon_i}{KT}+\frac{E_F}{KT}}+1}$$

$E_F = KT_F$

$$\boxed{T_F = \frac{E_F}{K} = \frac{1}{2m}\frac{(3\pi^2\rho)^{2/3}}{K}}$$

Fermi Temperature

— Above Approximation is good if $\boxed{T \ll T_F}$ — Degenerate Fermi Gas $\quad T \to 0$

$\to$ e — in solids —

⇒ Degeneracy Pressure

can not settle into any state since all states are occupied

— Creates an Illusion of a Pressure

Degeneracy Pressure
Fermi

⇒ Calculation of Degeneracy Pressure

— g – order of degeneracy $\quad g = 2s+1 \quad \boxed{g = 2}$

— Total Energy of Fermi Particles in the Volume $\qquad \sqrt{m^2+p^2}$

$$E_{TOT} = gV\int \langle n_i\rangle \varepsilon_i \frac{d^3p}{(2\pi)^3} = gV\int_0^{p_F} \frac{p^2}{2m}\frac{p^2\,d\Omega}{(2\pi)^3} = //$$

$$// \frac{gV4\pi}{2m(2\pi)^3}\int_0^{p_F} p^4\,dp = \frac{V}{10m\pi^2}p_F^5 = E_{TOT}$$

— Energy Density $\quad \frac{E_{TOT}}{V} = \boxed{e = \frac{1}{10m\pi^2}p_F^5}$

$$p_F = (3\pi^2\rho_N)^{1/3}$$

$$\boxed{e = \frac{1}{10m\pi^2}(3\pi^2)^{5/3}\rho_N^{5/3}}$$

— Analysis $\quad \boxed{e = \frac{1}{10m\pi^2}(3\pi^2)^{5/3}V^{-5/3}N^{5/3}} \qquad \boxed{\rho_N = \frac{N}{V}}$

For Fixed $N \quad e\uparrow \sim V^{-5/3}\;\; \boxed{V\downarrow}, \quad E_{TOT}\uparrow \sim V^{-2/3}$

(star shrinks by gravity $\quad E_{TOT}$ – increases)

$\boxed{N = fix}$

— For Fixed Number of Particles and $T \quad \boxed{F = E_{TOT} = E}$ free energy

$$\boxed{P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}} = -\frac{\partial E_{TOT}}{\partial V} = -\frac{\partial(eV)}{\partial V} = -e - \frac{\partial e}{\partial V}\cdot V = //$$

$$// = -e - V\frac{\partial e}{\partial \rho}\frac{\partial \rho}{\partial V} = -e + V\frac{\rho}{V}\cdot\frac{e\cdot\frac{5}{3}}{\rho} = -e + \frac{5}{3}e = \boxed{\frac{2}{3}e} = ?$$

$$\frac{\partial \rho_N}{\partial V} = \frac{\partial N/V}{\partial V} = -\frac{N}{V^2} = -\frac{\rho_N}{V}$$

$$\frac{\partial e}{\partial \rho} = \frac{\partial}{\partial \rho}\left(\frac{1}{10m\pi^2}(3\pi^2)^{5/3}\rho^{5/3}\right) = \frac{(3\pi^2)^{5/3}}{10m\pi^2}\rho^{2/3}\frac{5}{3} = \frac{e\cdot\frac{5}{3}}{\rho} =$$

— Thus $\quad \boxed{P} = \frac{2}{3}e = \frac{2}{3}\frac{(3\pi^2)^{5/3}}{10m\pi^2}\rho^{5/3} = \frac{(3\pi^2)^{5/3}}{15m\pi^2}\rho_N^{5/3} =$

using $\boxed{p_F = (3\pi^2\rho)^{1/3}} \quad // = \boxed{\frac{1}{15m\pi^2}p_F^5}$

HW – Copper at Room Temperature. $\varepsilon_F = 7.03\,eV$, $g_N = 6.3\times10^{8}\,(eV)^3 \sim \frac{1}{V}$

– $T_F = 81500$ deg,

$T_{Room} = 280° << T_F$ – Copper is Degenerate

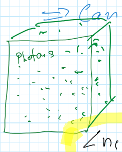
– $P = 3.78\times10^{5}$ atmospheres

⇒ Planck Formula

⇒ Electromagnetic Radiation in QM is Radiation of Photons

⇒ Photons are spin – 1 particles with rest mass = 0

⇒ Can Use Distribution Function For Bosons



$$\langle n_i\rangle = \frac{1}{e^{\alpha+\beta\varepsilon_i}-1}, \quad \alpha = -\mu\beta, \quad \beta = \frac{1}{KT}$$

$\varepsilon < \mu$

$$\boxed{\langle n_i\rangle = \frac{1}{e^{\frac{1}{KT}(\varepsilon_i-\mu)}-1}}$$

if $T \to 0$: $\frac{1}{KT}(\varepsilon_i - \mu) \to 0$

$e^{-\frac{|\varepsilon_i-\mu|}{KT}} \xrightarrow[T\to 0]{} 0$ not 0 $T \to 0$

$\varepsilon < \mu$: $\langle n_i\rangle \to -1$

$n_i \to \infty$, $\varepsilon = \mu$

$>$ $\varepsilon_i = \mu$ | at $T \to 0$

$$\boxed{\langle n_i\rangle = \frac{1}{e^{\frac{\varepsilon}{KT}}-1}}$$ photons

– Since photon mass = 0: $\mu = 0$

$\varepsilon_i \to 0$, $\boxed{\mu = 0}$

$\mu = m$, $\mu = m_0$

– Number of states for Photons

$$\boxed{N = \sqrt{g}\int \langle n_i\rangle \frac{d^3p}{(2\pi\hbar)^3}}$$

using $\boxed{P = \varepsilon_\gamma / c}$

$$dn_r = \sqrt{g}\frac{d^3p}{(2\pi\hbar)^3} = gV\frac{p^2\,dp\,d\Omega}{(2\pi\hbar)^3} = gV\frac{\varepsilon^2\,d\varepsilon\,d\Omega}{c^3(2\pi\hbar)^3} = gV\frac{\varepsilon^2\,d\varepsilon\,d\Omega}{c^3h^3}$$

h, $\hbar = \frac{h}{2\pi}$

– using $\varepsilon = h\nu = \hbar\omega$

$$dn_r = gV\frac{\nu^2\,d\nu\,d\Omega}{c^3}$$

$g = 2$ number of photon polarization

$S = 1$, $g = 3 \Rightarrow 2$

– $dn_\gamma = 2\cdot 4\pi\, V\frac{\nu^2 d\nu}{c^3}$ $\varepsilon = h\nu$

$$\varepsilon_{TOT} = \int \langle n_i\rangle\,\varepsilon_i\,dn_\gamma = V\int \frac{1}{e^{\frac{\varepsilon}{KT}}-1}\cdot\varepsilon_i\,\frac{8\pi\nu^2 d\nu}{c^3}$$

$\varepsilon = h\nu$

$$\varepsilon_{Tot} = \frac{8\pi V}{c^3}\int \frac{1}{e^{\frac{h\nu}{KT}}-1}\,h\nu\cdot\nu^2 d\nu$$

$$\varepsilon(\nu)\,d\nu = \frac{8\pi h V}{c^3}\,\frac{\nu^3\,d\nu}{e^{\frac{h\nu}{KT}}-1}$$

$$\boxed{u\,d\nu} = \frac{\varepsilon(\nu)}{V}\,d\nu = \frac{8\pi h}{c^3}\,\frac{\nu^3}{e^{\frac{h\nu}{KT}}-1}\,d\nu$$

$$\boxed{u(\nu) = \frac{8\pi h}{c^3}\,\frac{\nu^3}{e^{\frac{h\nu}{KT}}-1}}$$

Planck Formula