L7

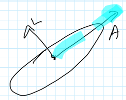
Wednesday, February 10, 2021 4:55 PM

Hydrogen Like Atoms

$\hat{H} = \frac{p^2}{2m} - \frac{Ze^2}{r}$

$e^2 = \alpha \hbar c$, $\alpha = \frac{1}{137.030}$

$L_i = \frac{(r \times P)_i}{\hbar}$



$[L_i L_j] = i \sum_k \varepsilon_{ijk} \hat{L}_k$

$\hat{A}_i = \frac{1}{2}(\hbar L \times P)_i - \frac{1}{2}(P \times \hbar L)_i + Ze^2 m \frac{r_i}{r}$

$[H L_i] = 0$, $[H L^2] = 0$, $[L_i L_j] = i \sum \varepsilon_{ijk} L_k$, $[L^2 L_i] = 0$

$[H A_i] = 0$, $[H A^2] = 0$, $[A_i A_j] = -2i m \hbar^2 H \sum_k \varepsilon_{ijk} L_k$, $[A^2, A] = 0$

$[L_i A_j] = i \sum_k \varepsilon_{ijk} A_k$

$[L_z A_z] = i \sum_k \varepsilon_{zzk} A_k = 0$

$[L^2, A^2] = 0$

$[L_z, A^2] = 0$

$[L^2, A_z] \neq 0$

$L^2 \to \ell(\ell+1)$

$\ell = ?$

$\ell = m_{max}$

$L_z \to m$

$\Psi_{E \ell m, \lambda_{A^2}, \lambda_{A_z}}(r, \theta, \varphi)$

$\Psi_{E, j m_j k, m_k}(\vec{r})$

$\hat{H} \Psi_{E \ell m \lambda_{A^2}, \lambda_{A_z}}(r, \theta, \varphi) = E \Psi_{E \ell m, \lambda_{A^2}, \lambda_{A_z}}(r, \theta, \varphi)$

$\hat{A}^2 = 2m \hat{H}(\hbar^2 L^2 + \hbar^2) + m^2 Z^2 e^4$

$\Rightarrow$ Redefinition.

$A_i = \sqrt{-2mH\hbar^2} \cdot M_i$!!

$[A_i A_j] = -2m\hbar^2 H i \sum \varepsilon_{ijk} L_k$

$-2mH\hbar^2 \, [M_i M_j] = -i 2m\hbar^2 H \sum \varepsilon_{ijk} L_k$

$[M_i M_j] = i \sum \varepsilon_{ijk} L_k$

$[L_i M_j] = i \sum \varepsilon_{ijk} M_k$

$[L_i L_j] = i \sum \varepsilon_{ijk} L_k$

Define

$\hat{J}_i = \frac{\hat{L}_i + \hat{M}_i}{2}$

$\hat{K}_i = \frac{\hat{L}_i - \hat{M}_i}{2}$

$J^2 = \frac{L_i + \hat{L}_i \hat{M}_i + \hat{M}_i \hat{L}_i + M_i^2}{4}$

$K^2 = \frac{L_i - L_i M_i - M_i L_i + M_i^2}{4}$

$L_i M_i + M_i L_i = 0$

$\hat{L}_i \hat{A}_i + \hat{A}_i \hat{L}_i = 0$

$\hat{J}^2 = \hat{K}^2$

$\hat{L}_i = \hat{J}_i + \hat{K}_i$

$\hat{M}_i = \hat{J}_i - \hat{K}_i$

$L^2 + M^2 =$

$(J+K)^2 + (J-K)^2 =$

$= J^2 + K^2 + \cancel{JK} + \cancel{KJ}$

$J^2 + K^2 - \cancel{JK} - \cancel{KJ} =$

$= 2J^2 + 2K^2$

$= 2(J^2 + K^2) -$

$= 2(J^2 + J^2)$

$= 4J^2$

$[(J_i - K_i), (J_j - K_j)] = i \sum_k \varepsilon_{ijk} L_k = i \sum_k \varepsilon_{ijk}(J_k + K_k)$

$[(J_i + K_i), (J_j - K_j)] = i \sum \varepsilon_{ijk} M_k = i \sum_k \varepsilon_{ijk}(\hat{J}_k - \hat{K}_k)$

$[(J_i + K_i), (J_j + K_j)] = i \sum \varepsilon_{ijk}(J_k + K_k)$

Solution

$[J^2, J_i] = 0$

$[K^2, K_i] = 0$

a) $[J_i J_j] = i \sum \varepsilon_{ijk} J_k$

b) $[K_i K_j] = i \sum_k \varepsilon_{ijk} K_k$

$[J_i K_j] = 0$

Angular Momentum

$[L_i L_j] = i \sum \varepsilon_{ijk} L_k$

$[L^2, L_i] = 0$

$L_z \to m$

$L^2 \to \ell = m^{max}$

$-\ell \leq m \leq \ell$

a) $J_z \to m_j$

$J^2 \to j = m_j^{max}$

$j(j+1)$

b) $K_z \to m_K$

$K^2 \to K = m_K^{max}$

$K(K+1)$

$-j \leq m_j \leq j \to (2j+1)$ m_j states for m_j

$-K \leq m_K \leq K \to (2K+1)$ m_K

$\Psi_{E, J m_j, K, m_K}$

$J = K = j$

$(2j+1)(2K+1)$ total Number of m_j, m_K

$(2j+1)^2$ states with same E

n^2

$J^2 = K^2$

$J = K$

$\to$ Introduce ... $\to n$ ...

$\Rightarrow \boxed{n = 2j+1}$

Calculation of Energy

$$\Rightarrow \boxed{A^2 = 2mH(\hbar^2 + \hbar^2 L^2) + m^2 z^2 e^4}$$

$$A^2 \Psi_{E j m j K m_K} = \left(2mH\,(L^2+1)\hbar^2 + m^2 z^2 e^4\right)\Psi_{E j m j K m_K}$$

$$-2mH\hbar^2 M^2 \Psi_{E j m K m_K} = \left(2mH(L^2+1)\hbar^2 + m^2 z^2 e^4\right)\Psi_{E j m j K m}$$

$$-2m\hbar^2 M^2 H \Psi \ldots = \left(2m(L^2+1)\hbar^2 H + m^2 z^2 e^4\right)\Psi \ldots$$

$$\boxed{-2m\hbar^2 M^2 E\,\Psi = \left(2m(L^2+1)\hbar^2 E + m^2 z^2 e^4\right)\Psi}$$

$$-m^2 z^2 e^4 \cancel{\Psi} = 2mE\hbar^2(M^2 + L^2 + 1)\,\Psi = 2mE\hbar^2(4J^2+1)\Psi \ldots$$
$$2mE\hbar^2(4j(j+1)+1)\Psi$$

$\leftarrow M^2 + L^2 = 4J^2$

$$\boxed{J^2 \Psi_{E,j,m_j,k,m_k} = j(j+1)\Psi \ldots}$$

$$-m^2 z^2 e^4 = 2mE\hbar^2 \underbrace{(4j^2+4j+1)}_{(2j+1)^2 = n^2}$$

$$-m^2 z^2 e^4 = 2mE\hbar^2 n^2 \qquad \boxed{e^4 = \alpha^2 \hbar^2 c^2}$$

$$E = -\frac{m^2 z^2 e^4}{2m\hbar^2 n^2} = -\frac{m z^2 e^4}{2\hbar^2 n^2} = -\frac{m z^2 \alpha^2 \hbar^2 c^2}{2\hbar^2 n^2}$$

$$\Rightarrow \boxed{E = -\frac{z^2 \alpha^2 m c^2}{2n^2}}$$

$\hookrightarrow n = 1$

$n = 2j + 1$

$n - 1$

$j(j+1)$ — $J^2 = K^2$ — $k(k+1)$

$j = k$

$\boxed{L = J + K}$

$\boxed{L_z = J_z + K_z}$

$m_{max} = \ell \qquad m_j^{max} + m_k^{max} \qquad 2j$

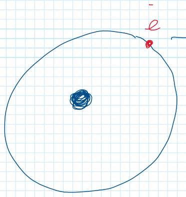
$j + k$

$-\ell \le m \le \ell$

$\ell \le 2j = n - 1$

$\boxed{\ell \le n-1}$

$\boxed{\ell \le n-1}$



e — E, n, ℓ

$n = 1, \quad S \to \ell = 0 \qquad (\ell \le 1-1)$

$n = 2; \quad S \to \ell = 0, \quad P \to \ell = 1 \qquad (\ell \le 2-1 \le 1)$

$? \quad \cancel{X} \to \cancel{\ell = 2}$

$\ell \le 3-1 = 2$

$\boxed{n = 3} \quad S, P, d, \cancel{f} \quad ?$

$\ell = 0$ — S

$\ell = 1$ = — P-state

$\ell = 2$ = d-state

$\ell = 3$ = f-state

Periodic Table $11\cancel{8} = z$. Largest $z = \boxed{123}$

z - - - z_{max}

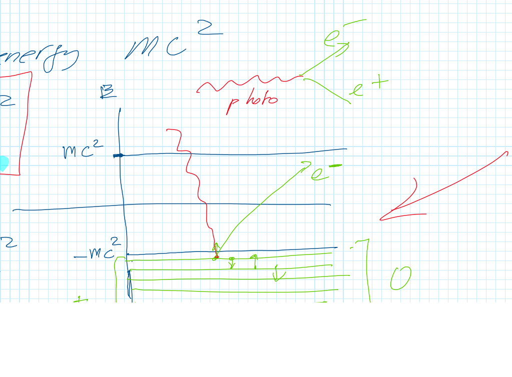
$$E_B = -\frac{z^2 \alpha^2 m c^2}{2}$$

e — rest mass $m \longrightarrow$ rest energy mc^2

$$\boxed{E_{TOT} = mc^2 - \frac{z^2 \alpha^2 m c^2}{2}}$$

minimal possible

$$-mc^2 = mc^2 - \frac{z_{max}^2 \alpha^2 m c^2}{2}$$



$$\frac{Z_{max}^2 \alpha^2 mc^2}{2} = 2mc^2$$

$$Z_{max}^2 \alpha^2 \cancel{mc^2} = 4 \cancel{mc^2}$$

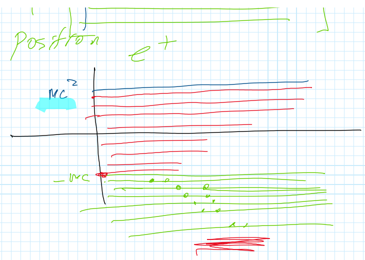
$$Z_{max}^2 \alpha^2 = 4$$

$$Z_{max}^2 = \frac{4}{\alpha^2}$$

$$Z_{max} = \frac{2}{\alpha} = 2 \times 137$$

$$\boxed{Z_{max} = 274}$$

Supercharged Nuclei



Finite Size 170
Z

Lead 208, 82, 120

Bismuth 209 83

Uranium 238

Oganesson $Z = 118$ $A = 295.2$

Au 79 197