Project

1. (50 points) From Normalization condition of Spherical Functions

$$\int |Y_l^m(\theta, \phi)|^2 d\Omega = 1$$

show that

$$Y_l^m(\theta, \phi) = (-1)^{\frac{m+|m|}{2}} \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos\theta)\, e^{im\phi}$$

2. (50 points) Calculate $[L_i\, A_j]$

3. (70 points) Calculate $[A_i\, A_j]$

4. (80 points) Show that $[H, A_i] = 0$.

To derive it first obtain th efollowing general *relations*:

$$[p_i, r^n] = -in\, r^{n-2} r_i$$

$$[p^2, r^n] = -in\,(p\cdot r)\, r^{n-2} - in\, r^{n-2}(r\cdot p)$$

$$\left[\frac{1}{r}, (L\times p)_i\right] = i\left(p_i r^2 - r_i (p\cdot r)\right)\frac{1}{r^3}$$

$$-\left[\frac{1}{r}, (p\times L)_i\right] = \left[\frac{1}{r}, (L\times P)^{\dagger}{}_i\right] = -\left[\frac{1}{r}, (L\times P)_i\right]^{\dagger} =$$

$$i\,\frac{1}{r^3}\left(r^2 p_i - (r\cdot p)\, r_i\right)$$

5. (100 points) Calculate $\hat{A}^2$

To check the eintermediate *steps* :

$$(p\times L)\,(p\times L) = [(L\times p)\,(L\times p)]^{+} = L^2\,p^2$$

$$(L\times p)\,(p\times L) = -L^2\,p^2$$

$$(p\times L)\,(L\times p) = -4\,p^2 - L^2\,p^2$$

$$\frac{\hbar^2}{4}\,(L\times p - p\times L)^2 = \hbar^2\,L^2\,p^2 + \hbar^2\,p^2$$

$$\overrightarrow{(L\times p)}\,\vec{r}\,\frac{1}{r} = -\hbar\,\frac{L^2}{r}$$

$$\vec{r}\,\frac{1}{r}\,\overrightarrow{(p\times L)} = \hbar\,\frac{L^2}{r}$$

$$\overrightarrow{(p\times L)}\,\vec{r}\,\frac{1}{r} = -\left(\overrightarrow{(L\times p)}\,\vec{r}\,\frac{1}{r}\right)^{\dagger} = \hbar\,\frac{L^2}{r} + 2\,i\,\vec{p}\,\vec{r}\,\frac{1}{r}$$

$$\vec{r}\,\frac{1}{r}\,\overrightarrow{(L\times p)} = -\left(\overrightarrow{(p\times L)}\,\vec{r}\,\frac{1}{r}\right)^{\dagger} = -\hbar\,\frac{L^2}{r} + 2\,i\,\vec{r}\,\frac{1}{r}\,\vec{p}$$