Lecture 1: Classical Mechanics

August 26, 2019

1 Introduction

- Scientific Truth
  - Math and Reality
  - for classical reality: We are wired for 3d + t - classical world
  - for quantum reality: need Math, and get use of it

Classical Mechanics

Galileo Galilei 15 Feb - 1564 - 8-Jan 1642

- Added to the Science the idea of measurement
- Made precise definition of measurable quantities such as displacement, time, velocity, acceleration - observables which can be measured
- First idea of inertial reference frame
- Kinematics

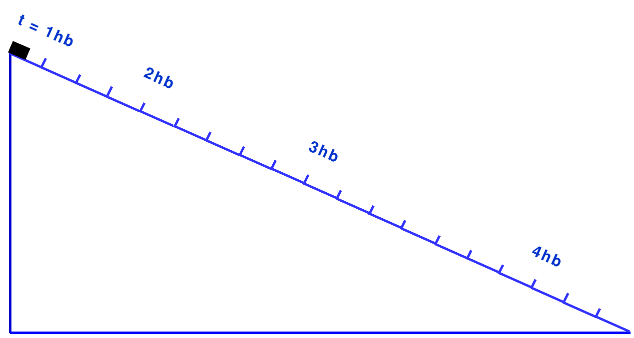

Figure 1: Galileo's Incline.

Johannes Kepler, 27 Dec 1579 - 15 Nov 1630

Kepler's three laws

- (1) All planets go by ellipse and sun is at one of the centers (focal point)
- (2) Equal times swipe equal areas

- (3) $T^2 = CR^3$ and C is the same for all planets

- Jordanus de Nemore (12-13th AD) - introduced the concept of the **Force**

- Isaac Newton 25 Dec 1642 - 20 May 1727
- Dynamics - based on three laws
- (1) Law of Intertia
- (2) $F = m \cdot a$
- (3) Action - Reaction Law
Development of Calculus
Isaac Newton, Gottfried Leibniz (1 Jan 1646 - Nov 14 1716 (age 70))
- Birth of Theoretical Mechanics
- Joseph - Louis Lagrange ((Turin) 1736 - (Paris) 1813) Euler's friend
Mechanique Analitique

2 Rules v.s. Principles

2.1 Advent of the Classical Mechanics

Laws and Laws everywhere
Isaac Newton (1742 - 1725)
Newton's Laws

1) Principle of Inertia

2) $\vec{F} = m \cdot \vec{a} = \frac{d\vec{p}}{dt}$

3) $\vec{F}_{12} = -\vec{F}_{21}$

Allows to obtain *Equation of Motion*

$$\ddot{\vec{r}} = \vec{a} = \frac{\vec{F}}{m} \tag{1}$$

$$\vec{v}(t) = \int_{t_0}^{t} \frac{\vec{F}}{m} dt \tag{2}$$

$$\vec{r}(t) = \int_{t_0}^{t} \vec{v}(t) dt \tag{3}$$

and two initial conditions are needed to solve above system of the equations

Work and Energy

$$W = \int_1^2 \vec{F} \cdot \vec{ds} = K_2 - K_1 \quad \text{where} \quad K = \frac{mv^2}{2} \quad \text{or} \quad K = T = \frac{1}{2} \sum m_n \dot{r}_n^2 \tag{4}$$

For *potential* forces one can define *Potential Energy* (U) and

$$W = \int_1^2 \vec{F} \cdot \vec{ds} = U_1 - U_2 \tag{5}$$

which results to the **law** of the *Conservation of Mechanical Energy*

$$\boxed{K_1 + U_1 = K_2 + U_2}$$

The potential energy U allows to define the force through the relation

$$-du = \vec{F} \cdot d\vec{r} = F_x dx + F_y d_y + F_z dz \tag{6}$$

which yields

$$F_x = -\frac{\partial U}{\partial x}, \quad F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z}, \tag{7}$$

Momentum

Momentum of one particle $\vec{p} = m\vec{v}$. Newton's 2nd law in it's original form was

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{8}$$

Momentum of the system of particles:

$$\vec{P}_{sys} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i \tag{9}$$

For closed systems:

$$\frac{d\vec{P}_{sys}}{dt} = \frac{d \sum_i \vec{p}_i}{dt} = \sum_i \vec{F}_i = 0 \tag{10}$$

That is $\boxed{\vec{P}_{sys} = const}$

Angular Momentum

For rotational motion one defines the **angular momentum**:

$$\vec{L} = I\vec{\omega} \tag{11}$$

with the "Newton 2nd Law"

$$\vec{\tau} = \frac{d\vec{L}}{dt} \tag{12}$$

For closed systems with net external torque being zero one obtains:

That is $\boxed{\vec{L}_{sys} = const}$.

We can continue discussing the many other laws of classical mechanics but the question is how these laws come to the existences. And what is fundamental: law's or principles which generate these laws?

2.2 Theoretical Mechanics

These questions were asked already in 18th century which resulted in the introduction of the theoretical Mechanics whose premise is that the fundamental are not the physical laws but the principles.

One of such principles being Hamilton's principle:

Lagrangian

One can reformulate whole classical mechanics by stating that any physical system can be described by fundamental Lagrangian[1], which for simple case of the collection of interacting particles can be expressed as

$$L = T - V \tag{13}$$

where T characterizes the kinetic energy of the system:

$$T = \frac{1}{2}\sum_{n,i} m_n \dot{r}_{n,i}^2 \tag{14}$$

and function V potential energy of the mutual interactions between particles in the system which has a local property (i.e. does not depend on velocities).

$$V \equiv V(r_{n,i}, t) \tag{15}$$

Thus

$$L \equiv L(r_{n,i}, \dot{r}_{n,i}, t) \tag{16}$$

and as such it is considered to be a function of independent variables of coordinates, $r_{n,i}$ and velocities $\dot{r}_{n,i}$.

Now we see that only one function (Lagrangian) is needed to define from Eq.(13) forces

$$F_{n,i} = \frac{\partial L}{\partial r_{n,i}} = -\frac{\partial V(r_{n,i})}{\partial r_{n,i}} \tag{17}$$

[1] Introduced by Joseph Lois Lagrange 25-jan 1736 - 10-Apr 1813

and momenta

$$p_{n,i} = \frac{\partial L}{\partial \dot{r}_{n,i}} = \sum_n m_n \dot{r}_{n,i} \tag{18}$$

Note that even though we found a way of obtaining forces and momenta from the one universal function, we don't arrive yet to the Neutron Law's (or dynamics) since the coordinates and velocities are considered independent variable.

To move beyond the definitions to dynamics that relates forces and momenta we need a *declaration of the principle.*

Backtrack discussion: To have an idea what kind of principle is required. We do a *reverse engineering*, by noticing that the cornerstone of the dynamics Newton's second law Eq.(8) can be formally presented through the only Lagrange function as:

$$\frac{d}{dt}\frac{\partial L}{\dot{r}_{n,k}} = \frac{\partial L}{\partial r_{n,i}} \tag{19}$$

Therefore the *principle* we are looking for should result the the above relation for the partial derivatives of the Lagrangian (referred as Langrage -Euler equation. Or in other words *The principle stipulates that the Classical world is realized in such a way that it relates $r_{n,i}$'s to $\dot{r}_{n,i}$'s through the relations of Eq.(19).*

2.3 The Genesis of Classical Mechanics: Hamilton's Principle

Now we can formulate the Classical Mechanics with just a few principles and show how all the physical laws follows:

1. The reality (the *building blocks* of reality) are coordinates r_k and velocities $\dot{r}_k$.
2. Any Physical System has its own Lagrangian which is function of independent variables of coordinates, velocities as well as time

$$L(r_n, \dot{r}_n, t) \tag{20}$$

3. The history/happening (which the beginning t_1 and the end time t_2) in the Physical System is characterized with the function of Action defined as

$$S = \int_{t_1}^{t_2} L(r_n, \dot{t}_n, t)dt \tag{21}$$

4. The classical reality ($r(t)$) has one way of realization, such way that S is extreemum, i.e.

$$\delta S = 0 \tag{22}$$

In the case of the considered mechanical system the "reality" is described by the path or trajectory of the motion $r(t)$. Therefore the statement of (3) can be restated as

3'. Among all possible paths , $r(t)$ only one is realized, $r_0(t)$ for which

$$\delta S = 0 \tag{23}$$

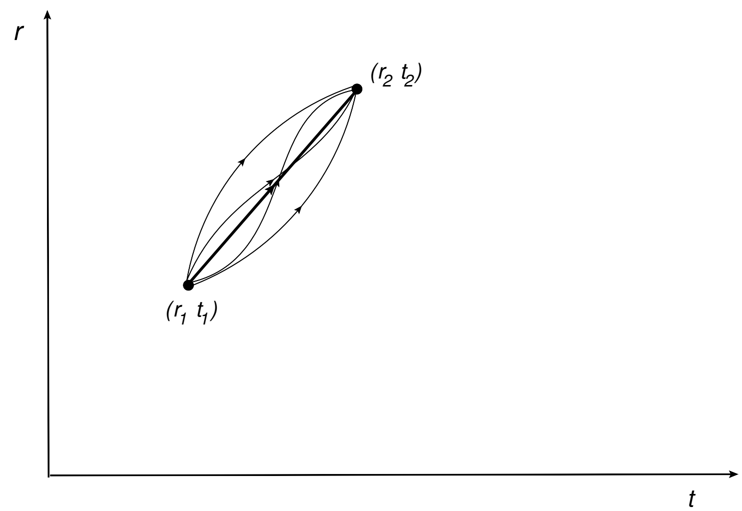

Figure 2: Different paths.

To present the mathematical form of the above statements let us consider the variation of $r_0(t)$ path to any arbitrary paths ($r(t) = r_0(t) + \delta r(t)$ such way that they originate from the same position at t_1 and converge to the same position at t_2 (see Fig.2). With such definition the mathematical version of the statement of (3') is:

$$\boxed{\frac{\delta S}{\delta r} = 0} \tag{24}$$

Stugel This relation is equivalent to the statement that all first order terms in δr should vanish in δS. For this we now calculate the action $S(t_1, t_2, r(t), \dot{r}(t))$ for arbitrary $r(t)$ and estimate it up to the linear order in $\delta r = r(t) - r_0(t)$. For this we estimate

$$S(t_1, t_2, r_0 + \delta r, \dot{r}_0 + \delta \dot{r}) = S_0 + \delta S = \int_{t_1}^{t_2} L(r_0 + \delta r, \dot{r}_0 + \delta \dot{r}, t) dt. \tag{25}$$

We first, considering r and $\dot{r}$ as independent variables, expand the Lagrange function:

$$L(r_0 + \delta r, \dot{r}_0 + \delta \dot{r}, t) = L(r_0, \dot{r}_0, t) + \sum_k \frac{\partial L}{\partial r_k} \delta r_k + \sum_k \frac{\partial L}{\partial \dot{r}_k} \delta \dot{r}_k \tag{26}$$

and insert it back to Eq.(25) which yields:

$$S_0 + \delta S = \int_{t_1}^{t_2} L(r_0, \dot{r}_0, t)dt + \sum_k \int_{t_1}^{t_2} \frac{\partial L}{\partial r_k}\delta r_k dt + \sum_k \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{r}_k}\delta \dot{r}_k dt \tag{27}$$

using the relation $\delta \dot{r}_k(t) = \frac{d}{dt}\delta r_k(t)$ in the above equation for δS one obtains:

$$\delta S = \sum_k \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial r_k}\delta r_k + \frac{\partial L}{\partial \dot{r}_k}\frac{d}{dt}\delta r_k \right) dt \tag{28}$$

For the second term of the RHS part of the equation, using integration by parts one can write

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{r}_k}(\frac{d}{dt}\delta r_k)dt = \frac{\partial L}{\partial \dot{r}_k}\delta r_k|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_k} \right) \delta r_k dt \tag{29}$$

where the first part of the RHS will be zero since the variation of the trajectory vanishes at the boundaries: $\delta r_k(t_1) = \delta r_k(t_2) = 0$. Inserting the remaining part of the RHS of Eq.(29) into Eq.(28) and applying the condition of the extremum of the action in Eq.(24) one obtains:

$$\delta S = \sum_k \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial r_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_k} \right) \delta r_k dt = 0. \tag{30}$$

Since the variation of δr_k was chosen arbitrarily, the above equation can be satisfied only if

$$\boxed{\frac{\partial L}{\partial r_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_k} = 0} \tag{31}$$

The above equation represents the Lagrange-Euler equation of motion.

Reparameterization Theorem

The Hamilton's principle and obtained equations survive any reparameterization of the form

$$q(r_k, \dot{r})_k) \quad \text{and} \quad \dot{q}(r_k, \dot{r})_k) \tag{32}$$

as far as the functional depencens does not have singularities. It results to the Langrange function: $L \equiv L(q_k(t), \dot{q}_k(t), t)$ which will satisfy Hamilton's principle as far as $L(r(t), \dot{r}(t), t)$ satisfies it, resulting to the new set's of Euler-Lagrange equation of motion:

$$\boxed{\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0} \tag{33}$$

Some Generalization: *All physical systems we know of can be written in therms of an action functional and a Langragian in a general case*

Example: Charged particle in an electromagnetic field
Introducing electromagnetic potentials: $\phi(r,t)$ and $\vec{A}(r,t)$ through which the magnetic and electric fields are defined as follows

$$B_k = (\nabla \times A)_k \quad \text{and} \quad E_k = \nabla_k \phi - \frac{1}{c}\frac{\partial A_k}{dt} \tag{34}$$

The Lagrangian for this system can be written as:

$$L = \sum_k \frac{1}{2} m \dot{r}_k^2 - q\phi(r,t) + \frac{q}{c}\vec{A}(r,t)\cdot\dot{\vec{r}} \tag{35}$$

where q is the charge of the particle.

Now we try to formally satisfy Eular-Lagrange equation by calculating:

$$\frac{\partial L}{\partial r_i} = -q\frac{\partial \phi(r,t)}{\partial r_i} + \frac{q}{c}\sum_j \dot{r}_j \frac{\partial A_j(r,t)}{\partial r_i} \tag{36}$$

and

$$\begin{aligned} \frac{\partial L}{\partial \dot{r}_i} &= m\dot{r}_i + \frac{q}{c}A_j \\ \frac{d}{dt}\frac{\partial L}{\partial \dot{r}_i} &= m\ddot{r}_i + \frac{q}{c}\sum_j \frac{\partial A_i}{\partial r_j}\dot{r}_j + \frac{q}{c}\frac{\partial A_i}{\partial t} \end{aligned} \tag{37}$$

Combining above to equations into the Euler-Lagrange equation (31) one obtains:

$$m\ddot{r}_i = -q\frac{\partial \phi}{\partial r_i} - \frac{q}{c}\frac{\partial A_i}{\partial t} + \frac{q}{c}\sum_j \left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot{r}_j \tag{38}$$

For the third term we can write

$$\frac{q}{c}\sum_j \left(\frac{\partial A_j}{\partial r_i} - \frac{\partial A_i}{\partial r_j}\right)\dot{r}_j = \frac{q}{c}\sum_{j,k}\epsilon_{kij}\dot{r}_j(\nabla \times A)_k = \frac{q}{c}(v \times B)_i \tag{39}$$

where we used the definition of the magnetic field, **B** from Eq.(34). The remaining two terms in Eq.(38) can be written using the definition of the electric field **E** from Eq.(34):

$$-q(\frac{\partial \phi}{\partial r_i} + \frac{1}{c}\frac{\partial A_i}{\partial t}) = qE_i. \tag{40}$$

Inserting Eqs.(39) and (40) into Eq.(38) we finally arrive at Lorenz force formula:

$$m\ddot{\mathbf{r}} = q(E + \frac{1}{c}(\mathbf{v} \times \mathbf{B})) \tag{41}$$

2.4 Some Properties of L

- **(A) Additivity:**: If A and B are closed non-interacting systems

$$L = L_A + L_B \tag{42}$$

and the equations of motions for individual systems are independent.
- **(B) Constant Factor:** Adding constant to L will not change the equation of the motion.
-**(C) Total Time Derivative:** Lagrangians differing by a function which can be represented as total time derivative correspond to the same equation of motion. Let us consider two Lagrangians L' and L such that

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} f(q, t) \tag{43}$$

then we observe that

$$S' = \int_{t_1}^{t_2} L'(q, \dot{q}, t) dt = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt + \int_{t_1}^{t_2} \frac{df(q, t)}{dt} dt = S + f(q^{(2)}, t_2) - f(q^{(1)}, t_1) \tag{44}$$

and thus S' and S differ by a constant factor. From which it follows that if $\delta S =$ then $\delta S' = 0$ resulting to the same equation of motion.

This property can be checked directly by considering the equation of motion of Eq.(33). Here one needs to show that:

$$\frac{\partial}{\partial q} \frac{d}{dt} f(q, t) - \frac{d}{dt} \frac{\partial}{\partial \dot{q}} \frac{d}{dt} f(q, t) = 0 \tag{45}$$

For this in the RHS part of the equation we use the following relation relation:

$$\frac{d}{dt} f(q, t) = \frac{\partial f(q, t)}{\partial q} \dot{q} + \frac{\partial f(q, t)}{\partial t} \tag{46}$$

and then

$$\frac{\partial}{\partial \dot{q}} \frac{d}{dt} f(q, t) = \frac{\partial}{\partial \dot{q}} \left(\frac{\partial f(q, t)}{\partial q} \dot{q} + \frac{\partial f(q, t)}{\partial t} \right) = \frac{\partial f(q, t)}{\partial q} \tag{47}$$

insetting this back to the RHS part of the Eq.(45) one obtains:

$$\frac{\partial}{\partial q} \frac{d}{dt} f(q, t) - \frac{d}{dt} \frac{\partial f(q, t)}{\partial q} = 0, \tag{48}$$

where one needs only to change the orders of the derivatives to prove the identity. Note that in both above derivations it was important that f was a function of q and t but not $\dot{q}$.

3 Reinventing Mechanics with New Insights in Known Laws

Now we can show how the known equations can be derived from the Lagrange function and "minimal action principle". It is worth to emphasize that even if we will rederive the known physical laws, this approach will give as new *insights* about the meaning of the considered laws:

3.1 Newton's Laws

Newton's First Law:
If no forces are acting on the particle and system does not depend explicitly on the position at which it was observed then: $\frac{\partial L}{\partial q} = 0$ and from Eq.(31 one obtains:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \tag{49}$$

from which it follows that $\frac{\partial L}{\partial \dot{q}}$ which for fixed mass particle represents the relation $\vec{v} = const$. Thus one observe that if the motion of the object does not depend on position explicitly its velocity will not change. This gives rather different interpretation to the law of Inertia.

Newton's Second Law:
Newton's second law follows from the Lagrange-Euler equation for the particle moving in the potential energy field with $L = T - V$. Thus the new *insight* we gain is that the motion in the case of the interact happens in such a way to keep the action at extreemum.

Newton's Third Law:
Let us consider interacting two particles in the closed system. In this case the Lagrangian of such a system can be written as:

$$L = \frac{1}{2}m_1\dot{r}_1^2 + \frac{1}{2}m_2\dot{r}_2^2 - V(|r_1 - r_2|). \tag{50}$$

The above defined Lagrangian clearly is invariant for space translation of $\vec{r_i} \to r_i + \epsilon$, $r = 1, 2$ Considering the variation of the L with respect to such a change of the coordinates one obtains:

$$\delta L = L(r_i + \epsilon) - L(r_i) = \sum_{i=1}^{2}\frac{\partial L}{\partial r_i}\delta r_i = \epsilon\sum_{i=1}^{2}\frac{\partial L}{\partial r_i} = 0. \tag{51}$$

Therefore one obtains:

$$\frac{\partial L}{\partial r_1} + \frac{\partial L}{\partial r_2} = 0 \quad \to \quad -\vec{F_1} - \vec{F_2} = 0 \quad \to \quad \boxed{\vec{F_1} = -\vec{F_1}}. \tag{52}$$

Thus Newton's third law states that if interaction between two particles are mutual and does not depend on the properties of the inertial space then the action and reaction forces are equal and opposite.

Energy Conservation and Uniformity of Time
For closed system the Lagrange function does not depend on time explicitly. Therefore

$$\frac{d}{dt}L = \sum_{i,n}\frac{\partial L}{\partial r_{n,i}}\dot{r}_{n,i} + \sum_{i,n}\frac{\partial L}{\partial \dot{r}_{n,i}}\ddot{r}_{n,i}. \tag{53}$$

Using now Lagrange equation Eq.(31) one replaces $\frac{\partial L}{\partial r_{n,i}}$ by $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{n,i}}$ in the above equation which results to:

$$\frac{d}{dt}L = \sum_{i,n}\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{n,i}} + \sum_{i,n}\frac{\partial L}{\partial \dot{r}_{n,i}}\ddot{r}_{n,i} = \sum_{i,n}\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}_{n,i}}\dot{r}_{n,i}) \tag{54}$$

Moving LHS part to the RHS part of the equation one obtains:

$$\frac{d}{dt}\left(\sum_{i,n}\dot{r}_{n,i}\frac{\partial L}{\partial \dot{r}_{n,i}} - L\right) = 0 \tag{55}$$

Introducing the energy of the system as:

$$E = \sum_{i,n}\dot{r}_{n,i}\frac{\partial L}{\partial \dot{r}_{n,i}} - L \tag{56}$$

we conclude that in the case when the Lagrangian does not explicitly depend of the time - which is equivalent of time flow to be uniform we have a conservation of energy. $\boxed{\text{E} = \text{const}}$.

Momentum Conservation and Uniformity of Space

Now we explore the property of the uniformity of space by considering an invariance of the Lagrangian with respect to the translation in space:

$$\vec{r}_n \to \vec{r}_n + \epsilon \tag{57}$$

where ϵ is a constant and n counts the number of the particles in the system. The change of the Lagrangean due to this transformation reads:

$$\delta L = \sum_{i,n}\frac{\partial L}{\partial r_{n,i}}\delta r_{n,i} = \epsilon\sum_{i,n}\frac{\partial L}{\partial r_{n,i}} = 0. \tag{58}$$

Since ϵ is an arbitrary constant one obtains:

$$\sum_{i,n}\frac{\partial L}{\partial r_{n,i}} = 0. \tag{59}$$

Now using the Langrage-Euler equation (31) one can write

$$\sum_{i,n}\frac{\partial L}{\partial r_{n,i}} = \sum_{i,n}\frac{d}{dt}\frac{\partial L}{\partial \dot{r}_{n,i}} = \frac{d}{dt}\sum_{i,n}\frac{\partial L}{\partial \dot{r}_{n,i}} = \frac{d}{dt}\sum_{n}\vec{p}_n = 0, \tag{60}$$

which results to the well known law of the conservation of momentum:

$$\sum_{n}\vec{p}_n = const \tag{61}$$

Angular Momentum Conservation and Isotropy of Time
Using the similar principle for isotropy of space. That is the Lagrangian will not change due the rotation one arrives to the conservation law for total angular momentum of the system

$$\sum_n \vec{L}_n = const \tag{62}$$

Galilean Relativity
How the Galilean relativity appears in the Lagrange formulation of Mechanics. Remind that the principle of Galilean Relativity is the requirement of the invariance of physical dynamics with respect to the following transformations:

$$\begin{aligned} \vec{r}'_n &= \vec{r}_n + \vec{V}_R t \\ t' &= t, \end{aligned} \tag{63}$$

where r_n, represent the coordinates of the n particles in the system and $\vec{V}_R$ is the velocity of the reference frame moving with respect to the frame in which the coordinates are defined by $\vec{r}_n$. Inserting these transformation into the Lagrange function one obtains:

$$L' = \sum_{n,i} \frac{m(\dot{r}_{n,i} + V_{R,i})^2}{2} - \sum_{n,i} V(r_{n,i} + V_{R,i}t) \tag{64}$$

Note that in many realistic cases for closed systems the potential energy V will not change at all, since it depends on the relative distances between the particles in the system. However one can prove the invariance for the most general case without invoking the latter property. For this we show that the modified Lagrangian in Eq.64 can be presented in the form:

$$L' = L + \frac{dF(r_n, t)}{dt}, \tag{65}$$

which according to the property of (C) discussed in Sec.2.4 will represent the same equation of the motion that the Lagrangian L represents.

To prove relation of Eq.(65) from Eq.(64) we show that for the kinetic energy part of the primes Lagrangian one obtains:

$$T' = \sum_{n,i} \left(\frac{m\dot{r}_{n,i}^2}{2} + m(\dot{r}_{n,i} V_{R,i}) + \frac{mV_{R,i}^2}{2} \right) = \sum_{n,i} \left(\frac{m\dot{r}_{n,i}^2}{2} + \frac{d}{dt} \left[m(r_{n,i} V_{R,i}) + \frac{mV_{R,i}^2 t}{2} \right] \right). \tag{66}$$

Now for the potential energy, we assume that it does no have singularities in the considerate range of the space and expand it as a sum of the Teylor series in $V_{R,i}t$:

$$V(r_{n,i} + V_{R,i}t) = V(r_{n,i}) + \sum_{k=1}^{\infty} \frac{1}{k!} \left[\frac{dV}{dr_{n,i}} \right]^k (V_{R,i}t)^k = V(r_{n,i} + \frac{d}{dt} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \left[\frac{dV}{dr_{n,i}} \right]^k (V_{R,i}) t^{k+1} \right), \tag{67}$$

where in the last part we collected the Teylor some into the form of the full time derivative.

Combining Eqs.(66) and (67) one can present the primed Lagrangian in the following of Eq.(65) with

$$F(r_{n,i},t) = \left[m(r_{n,i}V_{R,i}) + \frac{mV_{R,i}^2 t}{2}\right] + \sum_{k=1}^{\infty}\frac{1}{k!}\left[\frac{dV}{dr_{n,i}}\right]^k (V_{R,i})t^{k+1}, \tag{68}$$

which is function of the coordinates and time only. Thus the invariance is proved.

4 Generalization of Lagrangian and Lagrange Formalism

The generalization of the Lagrange formalism present a unique representation of the reality. It states that:

1. Any physical system can be characterized by Lagrangian (L), for which generalized coordinates (q_i) and velocities ($\dot{q}_i$) can be introduced.

2 Then the reality of this physical system corresponds to the extremity of this Lagrangian resulting to the generalized Lagrange-Euler equation:

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \tag{69}$$

3. Then we can introduce *Canonically Conjugate Momentum*:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{70}$$

and *Generalized Force*

$$F_i = \frac{\partial L}{\partial q_i}. \tag{71}$$

After which the generalized Lagrange- Euler equation reads as "old good Newton's Second Law":

$$\frac{dp_i}{dt} = F_i \tag{72}$$

4. The *Generalized Energy* of the system can be defined by the analogy:

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} - L \tag{73}$$

4.1 Example of Electromagnetic Interaction

Using the Lagrangian of electromagnetic interaction from Eq.(35) one can defined the generalized momentum as follows:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = mv_i + \frac{q}{c}\vec{A}(r,t)_i \tag{74}$$

Calculate also the generalized Force and Energy using the same Lagrangian

5 Our Place in the Reality

Some motions do not happen

6 Hamiltonian Formalism

In the above described Lagrangian formalism we assumed that reality is described on the basis of generalized coordinated q and velocities $\dot{q}$ with function $L(q, \dot{q}, t)$. However, alternatively we can describe the system through the coordinates q and momenta p.

The question which we will try to answer is that function instead of Lagrangian L will now describe the physical system.

6.1 Legandre Transformation

For this we first consider the general method of the change of conjugate variables: Consider function $f(x, y)$ that have full differential:

$$df = udx + vdy \quad \text{with} \quad u = \frac{\partial f}{\partial x} \quad \text{and} \quad v = \frac{\partial f}{\partial y}. \tag{75}$$

Change the independent variable from y to u. For this we define

$$g = u \cdot x - f(x, y). \tag{76}$$

Then for the full differential of function g one obtains:

$$dg = udx + xdu - df = udx + xdu - udx - vdy = -vdy + xdu \tag{77}$$

which now looks like a full differential of g as a function of y and u with

$$\frac{\partial g}{\partial y} = -v \quad \text{and} \quad \frac{\partial g}{\partial u} = x \tag{78}$$

6.2 The Hamiltonian

Using the above defined Legandre Transformations, we now defined Hamiltonian function as

$$H(q, p, , t) = \sum_i \dot{q}_i p_i - L(q, \dot{q}, t), \tag{79}$$

which is defined as a function of momenta, p_i, coordinates q_i and time, t. As a function of those variables for the full differential we can write

$$dH = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial t} dt. \tag{80}$$

One the other hand using definition of Eq.(79) we can write for the full differential also:

$$dH = \dot{q}dp + pd\dot{q} - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt = \dot{q}dp - \dot{p}dq - \frac{\partial L}{\partial t}dt, \tag{81}$$

where in the last part of the equation we used definition of the momentum $p = \frac{\partial L}{\partial \dot{q}}$ as well as Lagrange-Euler equation Eq.(refeq:LagEul) according to which $\frac{\partial L}{\partial q} = \dot{p}$.

Comparing Eqs.(80) and (81) one obtains the following set of equations, which are generally called *Hamilton's Canonical Equations*:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad -\dot{p}_i = \frac{\partial H}{\partial q_i}; \quad -\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}. \tag{82}$$

These equations are equivalent to Lagrange-Euler equations. While latter was one equation of second order, the Hamiltonian's canonical equations are two first-order differential equations.

We can generalize the above described procedure for Lagrange function of any dynamics system for which effective coordinates and velocities are defined. In this case the Hamiltonian strategy is to *make Legandre transformation from effective velocities to the effective momenta defined as a partial derivative of the Lagrangian with respect to the same velocities.*

6.3 Conservation Laws in the Hamiltonian Formalism

As it follows from Eq.(82) the conservation laws now are related to the absence of the explicit dependence of the given variables which are called *Cyclic* coordinates.

For example if given q_i does no appear in H then from Eq.(82) follows the corresponding momentum is conserved $\dot{p}_i = 0$.

The Hamiltonian formalism have some advantages compared to the Lagrange formal is: the order of differential equations are lower and conservation laws have more transparent interpretation. Thus in mechanics depending on the particulars of the problem one can choose either Lagrangian or Hamiltonian formlarims.

To see how the energy conservation appears in the Hamiltonian formalism, using Eq.(80) one obtains:

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}d\dot{q} + \frac{\partial H}{\partial p}d\dot{q} + \frac{\partial H}{\partial t}. \tag{83}$$

Now using expressions for $\dot{q}$ and $\dot{q}$ from the Hamilton's equation in Eq.(82) one obtains

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}. \tag{84}$$

Thus the energy conservation in the Hamiltonian formalism is related to the absence of the explicit time dependence in the Hamiltonian function H. This conservation is similar to the above discussed momentum conservation low, with t being considered as a cyclic variable for energy.

6.4 Time Evolution of the System and Poisson Brackets

If one now defines the mathematical operator (called Poisson Brackets) between two functions A and B in the form

$$\{A, B\} = \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right) \tag{85}$$

then we can claim that the time evolution of any observable $A(q, p, t)$ of considered system can be estimated through the Poission Brackets in the following form:

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}, \tag{86}$$

where H represents the Hamiltonian which governs the system. This can be easily proved by considering the full time differential of the function $A(q, p, t)$ and then using the Hamilton's Canonical Equations from Eq.(82). It is worth however mentioning that this framework is based on the assumption that for any physical system the building block of the description are coordinates, momenta and time. This assumption is foundation for the Hamiltonian formalism.

7 Additional New Insights

In the following two subsections we demonstrate how the Lagrange formalism allows to gain deeper insight in the workings of classical physics.

8 The Prejudice of the Inertial Reference Frame

Historically Classical Physics was developed on the paradigm that the true processes can be and should be observed in the reference frame which are at rest or moving with the constant velocity. This prejudice was so strong that Albert Einstein first needed to developed the Special Theory of Relativity before to move to its General version.

This prejudice was based on Newton's three laws that only in IRF's Newtonian dynamics in general and Newton's Second law in particular are meaningful. However Lagrange formalism indicates that the physical reality is described by Lagrange-Euler equation (31) with Newton's 2nd Law being one particular example.

We now show that Lagrange-Euler equations hold in any coordinate system (not only in IRFs). This follows from the Action Principle, *which is a statement about paths and not about coordinates.*

Let us consider the following coordinate transformations:

$$q_a = q_a(x_1, x_2, \cdots, x_3N, t) \tag{87}$$

with the t-dependence indicating on the possibility of using a new coordinate system that changes with the time, i.e.

$$\dot{q}_a \equiv \frac{dq_a}{dt} = \frac{\partial q_a}{\partial x^A} \dot{x}_A + \frac{\partial q_a}{\partial t}, \tag{88}$$

where A sums all the degrees of freedom in the coordinate system of x_i's.

Assuming that transformation of Eq.(87) is reversible, i.e. is continues and does not contain singularities, we should be able to inver the relationship of Eq.(88) in the form:

$$\dot{x}^A = \frac{\partial x^A}{\partial q_b}\dot{q}_b + \frac{\partial x^A}{\partial t}, \tag{89}$$

Our goal is not to check, if the Lagrange function in the coordinate system of x^A satisfies the Lagrange-Euler equation (that is the physical system is realized with dynamical equation of the motion) what happen to the same Lagrange function observed in the reference frame q_a?. Mathematically this means that if

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) = \frac{\partial L}{\partial x^A} \tag{90}$$

whether the same relation is true for any other reference frame q_a which is obtained from the original reference frame by the continuous transformation of Eq.87.

First, we calculate

$$\frac{\partial L}{\partial q_a} = \frac{\partial L}{\partial x^A}\frac{\partial x^A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}^A}\frac{\partial \dot{x}^A}{\partial q_a} = \frac{\partial L}{\partial x^A}\frac{\partial x^A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}^A}\left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x^A}{\partial q_a \partial t}\right), \tag{91}$$

where in the last part Eq.(89) is used to calculate $\frac{\partial \dot{x}^A}{\partial q_a}$.

Next we calculate:

$$\frac{\partial L}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{x}^A}\frac{\partial \dot{x}^A}{\partial \dot{q}_a} = \frac{\partial L}{\partial \dot{x}^A}\frac{\partial x^A}{\partial q_a} \tag{92}$$

where in the last part we again used Eq.(89) to calculate: $\frac{\partial \dot{x}^A}{\partial \dot{q}_a} = \frac{\partial x^A}{\partial q_a}$.

Now using Eq.(92) we calculate:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\frac{\partial x^A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}^A}\frac{d}{dt}\left(\frac{\partial x^A}{\partial q_a}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right)\frac{\partial x^A}{\partial q_a} + \frac{\partial L}{\partial \dot{x}^A}\left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x^A}{\partial q_a \partial t}\right), \tag{93}$$

where in the last step we used the relation: $\frac{d}{dt}\left(\frac{\partial x^A}{\partial q_a}\right) = \left(\frac{\partial^2 x^A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial^2 x^A}{\partial q_a \partial t}\right)$.

Now, subtracting Eq.(91) from (93) one obtains:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_a} - \frac{\partial L}{\partial q_a} = \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A}\right) - \frac{\partial L}{\partial x^A}\right]\frac{\partial x^A}{\partial q_a}. \tag{94}$$

The last relation shows that if Lagrange-Euler equation is satisfied in original reference frame x^A (i.e. $\frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = 0$) then the same equation is satisfied in the new reference frame q_a if $det(\frac{\partial x^A}{\partial q_a}) \neq 0$.

This we can interpret as road that we take as to the General Theory of Relativity, since if physical reality is reflected in Euler-Lagrange equation, then what we proved is that the equation is satisfied not only in the inertial reference frame but in any general frame that can be obtained through the nonsingular transformation from our reference frame.

8.1 Example: Rotating Coordinate System

This example also will elucidate the meaning of the force. According to Newtonian dynamics the force is meaningful only in the inertial reference frame in which it can be related to the cause-effect chain of the physical phenomena.

All effect which are due to the noninertiality of the reference frame is not considered a force. Such definition we will call Newtonian Force.

Let us consider now the Lagrangian of free particle in a current reference frame,

$$L = \frac{1}{2}m\dot{r}^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{95}$$

then consider a transformation do to moving to the rotating reference frame which results to:

$$\begin{aligned} x' &= x \cdot cos(\omega t) + y \cdot sin(\omega t) \\ y' &= y \cdot cos(\omega t) - x \cdot sin(\omega t) \\ z' &= z. \end{aligned} \tag{96}$$

Solving Eq.(96) with respect to r's one obtains:

$$\begin{aligned} x &= x' \cdot cos(\omega t) - y' \cdot sin(\omega t) \\ y &= x' \cdot sin(\omega t) + y' \cdot cos(\omega t) \\ z &= z'. \end{aligned} \tag{97}$$

Inserting Eq.(97) into Eq.(95) one obtains:

$$L = \frac{1}{2}m\left[(\dot{x}' - \omega y')^2 + (\dot{y}' + \omega x')^2 + \dot{z}^2\right] = \frac{1}{2}m(\vec{\dot{r}'} + \vec{\omega} \times \vec{r'})^2, \tag{98}$$

where in the second part of the above equation we introduced vector $\vec{\omega} = \omega \cdot \hat{n}_z$

Now let us use Lagrange-Euler equation considering first $\frac{\partial L}{\partial \dot{r}'_k}$, one obtains:

$$\frac{\partial L}{\partial \dot{r}'_k} = m(\dot{r}' + [\omega \times r'])_k \tag{99}$$

then $\frac{\partial L}{\partial r'_k}$, for which we obtain:

$$\frac{\partial L}{\partial r'_k} = m([\dot{r}' \times \omega] - [\omega \times [\omega \times r']])_k \tag{100}$$

Using Eq.(99 one obtains:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}'_k} = m(\ddot{r}' + [\omega \times \dot{r}'])_k \tag{101}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}'_k} - \frac{\partial L}{\partial r'_k} = m\left(\ddot{r}'_k + [\omega \times [\omega \times r']]_k + 2[\omega \times \dot{r}'])_k\right) = 0 \tag{102}$$

which can be presented in more familiar form:

$$m\ddot{r}'_k = m[[\omega \times r'] \times \omega]_k + 2m[\dot{r}' \times \omega]_k \equiv F_k^{CF} + F_k^{Cor} \tag{103}$$

where in the RHS we recognize the Centrifugal (F^{CF}) and Coriolis (F^{Cor}) forces

$$\begin{aligned} F_k^{CF} &= m[[\omega \times r'] \times \omega]_k \\ F_k^{Cor} &= 2m[\dot{r}' \times \omega]_k \end{aligned} \tag{104}$$

This example shows how the difference between Newtonian and non-Newtonian forces disappear in the Lagrange formulation of the mechanics with more fundamental meaning of the partial coordinate derivative of the L function: $\frac{\partial L}{\partial r_k}$.