Cosmo2m

Monday, December 3, 2018 12:27 AM

Cosmo2

Monday, November 19, 2018
2:18 PM

⇒ Again About FLRW Metric

⇒ Curved 3d Hypersphere is defined as

$$g_{ij}dx^i dx^j = \frac{dx^2}{\left(1+\frac{K}{4}r^2\right)^2} + \frac{dy^2}{\left(1+\frac{K}{4}r^2\right)^2} + \frac{dz^2}{\left(1+\frac{K}{4}r^2\right)^2}$$

In polar coordinates

$$g_{ij}\,dx^i dx^j = \frac{dr^2 + r^2 d\Omega^2}{\left(1+\frac{K}{4}r^2\right)^2}$$

⇒ Introducing $\rho = \frac{r}{\left(1+\frac{K}{4}r^2\right)^{-1}}$ and

$$r = \frac{2\rho}{1-\sqrt{K\rho^2}}$$

One obtains

$$g_{ij}dx^i dx^j = -\frac{d\rho^2}{1-K\rho^2} - \rho^2 d\Omega^2$$

⇒ This is pure geometry.

> Lets go Back to Schwartzschild Metric

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2GM}{c^2 r} & & & \\ & -\frac{1}{1-\frac{2GM}{c^2 r}}, & & \\ 0 & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

and consider the infinite medium with isotropy and uniformity [Cosmological Principle]

$M = \frac{4\pi}{3} r^3 \rho$ $\qquad \rho \neq r$

$$g_{\mu\nu} = \begin{pmatrix} 1-\frac{2G4\pi r^2}{c^2 3} & & & \\ & -\frac{1}{1-\frac{2G\frac{4\pi}{3}r^2}{c^2}}, & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}$$

define $K = \frac{8\pi G}{3c^2}\rho$

$$g_{\mu\nu} = \begin{pmatrix} (1-Kr^2) & & & \\ & -\frac{1}{1-Kr^2} & & \\ & & -r^2 & \\ & & & -r^2\sin^2\theta \end{pmatrix}$$

- r — distance from the arbitrary refference frame.

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and consider the infinite medium with Isotropy and uniformity [Cosmological Principle]

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define $K = \frac{8\pi}{3c^2} G\rho$

$$g_{\mu\nu} = \begin{pmatrix} (1 - Kr^2) & & & \\ & -\frac{1}{1 - Kr^2} & & \\ & & -r^2 & \\ & & & -r^2 \sin^2\theta \end{pmatrix}$$

- r – distance from the arbitrary reference frame.

$\Rightarrow$ define $\tau = \int \frac{dt}{a(t)} \qquad d\tau = \frac{dt}{a(t)}$

$$ds^2 = a(\tau)^2 \left(d\tau^2 c^2 - \left[\frac{dr^2}{1 - Kr^2} - r^2 d\Omega^2 \right] \right)$$

for $K = 0$ | $\rightarrow$ minkowski space

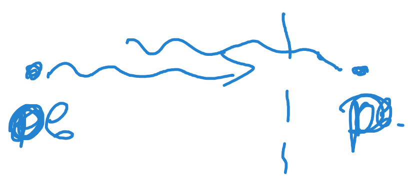
g_{00} $k=0$ ⟶ Minkowski space where we live

Conformal time ⟶ looks like our space at $k \to 0$

$\boxed{\text{⟹ But the cosmological time is } \hat{t}^{\beta}}$

– to expect the coordinates to range $-\infty$ + $+\infty$

– however $\tau \to -\infty$ $+\infty$ $a(t)$ can not be arbitrary depends on $a(t)$



assume universe begun at $\hat{t}=0$

– consider observer at $\hat{t}_{obs}$

if $\tau \to -\infty$ at $t \to 0$ observer will be able to receive all the signals from the past

$\Rightarrow$ if $a(t) = a_0 t^\alpha$

$$\tau = \frac{1}{a_0}\int \frac{dt}{t^\alpha} \sim \frac{1}{(\alpha-1)a_0}\frac{1}{t^{\alpha-1}}$$

if $\alpha \geq 1$ τ — diverges

if $\alpha < 1$ — there exists partial horizon

there is τ_{min}

$\Rightarrow$ Back to the Cosmological Eqs

$$\left(\frac{\dot a}{a}\right)^2 + \frac{Kc^2}{a^2} - \frac{\Lambda c^2}{3} = \frac{8\pi G}{3}\rho$$

$$\frac{2\ddot a}{a} + \left(\frac{\dot a}{a}\right)^2 + \frac{Kc^2}{a^2} - \Lambda c^2 = -\frac{8\pi G}{c^2}P$$

Redefine $\boxed{Kc^2 = \tilde K^2, \ \Lambda c^2 = \tilde\Lambda, \ \frac{G}{c^2}P = \tilde P}$ $G\rho = \tilde\rho$

$$\begin{cases} 3\frac{\dot a^2}{a^2} + \frac{3\tilde K^2}{a^2} - \tilde\Lambda = 8\pi\tilde\rho \\ -\frac{2\ddot a}{a} - \frac{\dot a^2}{a^2} - \frac{\tilde K}{a^2} + \tilde\Lambda = 8\pi\tilde P \end{cases}$$

(1) $\dot a^2 = \frac{8\pi\tilde\rho}{} + \frac{\tilde\Lambda}{3} - \frac{\tilde K}{a^2}$

$$a^2 \quad 3 \quad \quad a$$

$$-\frac{2\ddot{a}}{a} - \frac{8\pi\rho}{3} - \frac{\Lambda}{3} + \cancel{\frac{k}{a^2}} - \cancel{\frac{k}{a^2}} + \Lambda = 8\pi\tilde{P}$$

$$-\frac{\ddot{a}}{a} - \frac{4\pi}{3}\rho + \frac{1}{3}\Lambda = 4\pi\bar{P}$$

(2)' $$\boxed{\frac{\ddot{a}}{a} = -\frac{4\pi}{3}(\rho + 3P) + \frac{\Lambda}{3}}$$

From $\frac{d}{dt}$ (1) $$\frac{2\dot{a}\ddot{a}}{a^2} - \frac{2\dot{a}^3}{a^3} = \frac{8\pi\dot{\rho}}{3} + \frac{2k\dot{a}}{a^3}$$

(1') $$\frac{\ddot{a}}{a} = \frac{a}{\dot{a}}\frac{4\pi\dot{\rho}}{3} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}$$

$$(1') - (2) = \frac{a}{\dot{a}}\frac{4\pi}{3}\dot{\rho} + \frac{\dot{a}}{a^2} + \frac{k}{a^2} + \frac{4\pi}{3}(\rho + 3P) - \frac{\Lambda}{3} = 0$$

$\Rightarrow$

$$\dot{\rho} + \frac{3}{4\pi}\frac{\dot{a}}{a}\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3}\right) + \frac{\dot{a}}{a}(\rho + 3P)$$

(+)

(+) from (1) $$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} - \frac{\Lambda}{3} = \frac{8\pi\rho}{3}$$

$$\dot{\rho} + \frac{3}{4\pi}\frac{\dot{a}}{a}\frac{8\pi}{3}\rho + \frac{\dot{a}}{a}(\rho + 3P) = 0$$

$$\dot{\rho} + 2\frac{\dot{a}}{a}\rho + \frac{\dot{a}}{a}(\rho + 3P) = 0$$

$$\boxed{\dot{\tilde{\rho}} + 3\frac{\dot{a}}{a}(\tilde{\rho}+\tilde{P}) = 0}$$

Unknown a, $\rho(t)$ $P(t)$
underdetermined

Specify the matter content of the Universe $P = F(\varepsilon)$ – Equation of state of the Universe

⇒ Perfect Fluid $P = \text{Const} \cdot \varepsilon$
$\tilde{P} = w\,\tilde{\rho}$

→ Equation of state parameter $\frac{dP}{d\rho} = \sqrt{}$

if $w = 0$ Matter Dominated Universe

$w = \frac{1}{3}$ – Pure Radiation
Radiation Dominated Universe

– Standard Matter $0 \le w < 1$
– Can not be $1 \ge 1$

$w = 1$ stiff matter

No Standard Equation

Non standard eqn.

$-1 \leq w < -\frac{1}{3}$ – dark energy

$w = -1$ – cosmological const

$w \leq -1$ – phantom fluid.

for $\tilde{P} = w\tilde{\rho}$

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{a}}{a}(1+w)$$

$$\rho = \rho_0 a^{-3(1+w)}$$

4.2 Cosmological Solutions

4.2.1 Matter-Dominated Universe

$w = 0$ $\rho = \rho_0 a^{-3}$ $\propto K$

4.22
$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{K}{a^2}$$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi}{3}\frac{\rho_0}{a^3} + \frac{\Lambda}{3} - \frac{K}{a^2}$$

$$t - t_0 = \int \frac{1}{\left[\frac{8\pi\rho_0}{3a} + \frac{\Lambda}{3}a^2 - K\right]^{\frac{1}{2}}}\, da$$

— Case $k=0$ $\Lambda=0$

$$t-t_0=\int\left[\frac{8\pi\rho_0}{3a}\right]^{-\frac{1}{2}}da=\frac{1}{\sqrt{8\pi\rho_0/3}}\,\frac{2}{3}a^{3/2}$$

$$a(t)=(6\pi\rho_0)^{\frac{1}{3}}(t-t_0)^{\frac{2}{3}}$$

assume $a(t_0)=0$ / set $t_0=0$

$a(t)\sim t^{2/3}$ — particle Horizon

$\rho(t)\sim t^{-2}$ / $\rho(t)\big|_{t\to 0}\longrightarrow\infty$

Big Bang

⇒ Case $k=1$

$$t-t_0=\int\left[\frac{8\pi\rho_0}{3a}-1\right]^{-\frac{1}{2}}da=\int\frac{\sqrt{a}}{\sqrt{\frac{8\pi\rho_0}{3}-a}}\,da$$

$$a=\frac{8\pi\rho_0}{3}\sin^2\left(\frac{u}{2}\right)$$

$$t-t_0=\int\frac{8\pi\rho_0}{3}\sin^2\left(\frac{u}{2}\right)du=\frac{4\pi\rho_0}{3}(u-\sin u)$$

= no explicit solution for $a(t)$

Parametric form

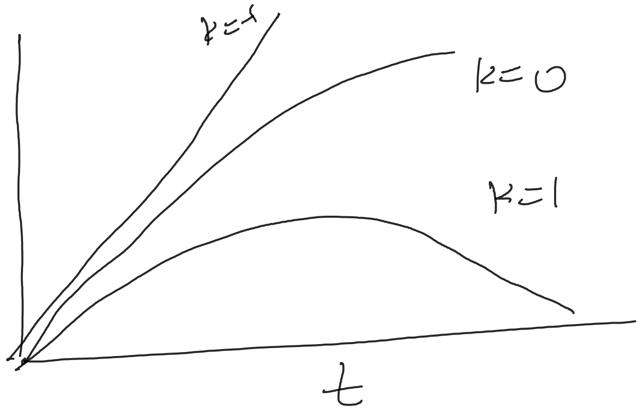
$$\left.\begin{aligned}a&=\frac{4\pi\rho_0}{3}(1-\cos u)\\ t&=\frac{4\pi\rho_0}{3}(u-\sin u)\end{aligned}\right]\text{cycloid}$$

2u4 $t_0=0$ $a(t=0)=0$

$$a_{max} = \frac{8\pi\rho_0}{3} \qquad t_{max} = \frac{4\pi^2\rho_0}{3}$$

$$\boxed{u=\pi}$$

$$t_{end} = \frac{8\pi^2\rho_0}{3} \to a=0$$



4.2.2. Radiation-dominated Universe

Restrict $\Lambda=0$

$$\frac{\dot\rho}{\rho} = -3\frac{\dot a}{a}(1+w) : \rho=\rho_0 a^{-3(1+w)}$$

$$\tilde P = \frac{1}{3}\tilde\rho \Rightarrow \boxed{\rho=\rho_0 a^{-4}}$$

$$\frac{\dot a^2}{a^2} = \frac{8\pi}{3}\rho_0 a^{-4} - \frac{k}{a^2}$$

$$\dot a^2 = \frac{8\pi}{3}\rho_0\frac{1}{a^2} - k$$

$$t-t_0 = \int\left[\frac{8\pi\rho_0}{3a^2}-k\right]^{-\frac{1}{2}}da = \int\frac{a\,da}{\sqrt{\frac{8\pi\rho_0}{3}-ka^2}}$$

$$t - t_0 = -\frac{1}{k}\sqrt{\frac{8\pi\rho_0}{3} - k a^2} \qquad k = \pm 1$$

for $k = 0$

$$t - t_0 = \left(\frac{8\pi\rho_0}{3}\right)^{-\frac{1}{2}} \frac{1}{2} a^2$$

Solving for a

$$a(t) = \begin{cases} \sqrt{2}\left(\frac{8\pi\rho_0}{3}\right)\sqrt{t} & \text{if } k = 0 \\ \sqrt{(8\pi\rho_0/3) - (t-t_0)^2} & \text{if } k = 1 \\ \sqrt{(t-t_0)^2 - \frac{8\pi\rho_0}{3}} & \text{if } k = -1 \end{cases}$$

for $k=0$ $a(t) \sim t^{1/2}$ $\ln\rho = -2\ln t$ $\rho(t) \sim \frac{1}{t^2}$

for $k=1$ universe is finite:

there exist a particle Horizon

Say at the beginning we have radiation dominated universe with some matter

$a(t) \sim \sqrt{t}$ $\rho_{rad} \sim \frac{1}{t^2}$ $\rho_{mat} \sim \frac{1}{a^3} \sim \frac{1}{t^{3/2}}$

at matter dominates

$$a(t) \sim t^{2/3} \qquad \rho_{mat} \sim \frac{1}{t^2} \qquad \rho_{rad} \sim \frac{1}{a^4} \sim t$$

4.2.3 The Einstein Static Universe

Consider
$$\frac{\dot{a}^2}{a^2} = \frac{8\pi\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2}$$

take $k=1$

Static Solution $\dot{a} = 0$

$$\frac{8\pi\rho_E}{3} + \frac{\Lambda}{3} - \frac{k}{a_E^2}$$

$$\frac{1}{a_E} = \sqrt{\frac{8\pi\rho_E}{3} + \frac{\Lambda}{3}}$$

— size of the universe is defined by Energy and Cosmological Const.

from (4.23) other Field Equation

$$0 = -\frac{4\pi}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad \Big| \; p = 0$$

$$\frac{4\pi}{3}\rho_E = \frac{\Lambda}{3} \qquad \Lambda = 4\pi\rho_E$$

Thus ρ and Λ are not independent

$\boxed{\Lambda \text{ should be } > 0}$

! Not stable for ρ-perturbations.

4.2.4 De Sitter Universe

- Consider $\Lambda \neq 0$ with $\rho = P = 0$
  No matter and radiation

(1)

$$\frac{\dot a^2}{a^2} = \frac{\Lambda}{3} - \frac{k}{a^2} \quad \Big| \quad \dot a^2 = \frac{a^2\Lambda}{3} - k$$

$$\dot a^2 = \frac{\Lambda}{3}\left(a^2 - \frac{3k}{\Lambda}\right)$$

$$\dot a = \sqrt{\frac{\Lambda}{3}}\sqrt{a^2 - 3k/\Lambda}$$

$$\sqrt{\frac{\Lambda}{3}}(t - t_0) = \int \frac{da}{\sqrt{a^2 - 3k/\Lambda}}$$

for $k = 0$ $\quad a(t) = a_0 e^{\sqrt{\Lambda/3}\cdot t}$

→ de Sitter Solution

No Big Bang

⇒ General Solution

$$a(t) = a_0 \cosh\left(\sqrt{\frac{\Lambda}{3}}\,t\right) + a_0\sqrt{1-\frac{3k}{\Lambda a_0^2}}\,\sinh\left(\sqrt{\frac{\Lambda}{3}}\,t\right)$$

$$a_0 = a(0)$$

→ resulting Line Element

$$ds^2 = -dt^2c^2 + a_0^2\, e^{2\sqrt{\frac{\Lambda}{3}}ct}\left(dx^2+dy^2+dz^2\right)$$

→ apparently non-static

→ Factually is static

Consider Time Translation

$$t \to t' + T \qquad T = \text{const}$$

$$ds \to ds'^2 = -dt'^2 + a_0^2\, e^{\sqrt{\frac{\Lambda}{3}}(t'+T)c}\,(dx^2+dy^2+dz^2)$$

→ rescaling $x' = e^{\sqrt{\frac{\Lambda}{3}}T}\,x$

$$ds'^2 = ds^2$$

⇒ Form invariant under the time Translation and hence static.