Lecture1

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Lagrangian, Gauge Invariance, QED

$$\boxed{\begin{aligned}\mathcal{L}(x) &= \mathcal{L}_0 + \mathcal{L}'(x) = \bar{\psi}(x)[\gamma_\mu(i\partial_\mu - eA_\mu) - m]\psi - \\ &\quad -\frac{1}{4}F_{\mu\nu}^2(x) \\ F_{\mu\nu}(x) &= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\end{aligned}}$$

$$\psi'(x) \rightarrow e^{i\alpha(x)}\psi(x)$$

$$A'_\mu(x) \rightarrow A_\mu + \frac{1}{q}\partial_\mu\alpha$$

– Introduce Covariant Derivative

$$\boxed{D_\mu \equiv \partial_\mu - ieA_\mu}$$

$$- D_\mu\psi \rightarrow e^{i\alpha(x)}D_\mu\psi$$

⟹ Some Notations

$$\partial_\mu = (\partial_0, -\vec{\nabla})$$

$$\gamma_\mu\partial_\mu = \gamma_0\partial_0 + \gamma_i\nabla_i$$

$\gamma_\mu \partial_\mu = \not\partial$,

$\gamma_\mu A_\mu = \not{A}_\mu$

$\gamma_0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma_i = \beta \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\bar{\psi} = \psi^+ \gamma_0$

$J_\mu = e \bar{\psi}(x) \gamma_\mu \psi$

↳ Transition Current

1.2 Equation of Motion

⇒ General Form of Equation of Motion

$$\boxed{\partial_\mu \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}} = \frac{\partial \mathcal{L}}{\partial \varphi}}$$

⇒ Considering for electromagnetic field

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial A^\nu_{,\mu}} = \frac{\partial \mathcal{L}}{\partial A^\nu} \quad (*)$$

$$\frac{\partial \mathcal{L}}{\partial A^\nu} = -\bar{\psi} e \gamma^\nu \psi = -J^\nu$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial A^\nu_{,\mu}} = -\frac{1}{4} \partial_\mu \left[\frac{\partial \left[(\partial_\sigma A_\delta - \partial_\delta A_\sigma) \right]^2}{\partial \partial_\mu A^\nu} \right] =$$

$$= -\frac{2}{4} \partial_\mu \left[\left(\delta^{\mu\sigma} \delta^{\delta\nu} - \delta^{\mu\delta} \delta^{\nu\sigma} \right) \partial_\sigma A^\delta - \partial_\delta A^\sigma \right)$$

$$= -\frac{1}{2} \partial_\mu \left[\partial_\mu A^\nu - \partial_\nu A^\mu - \partial_\nu A^\mu + \partial_\mu A^\nu \right)$$

$$= - \partial_\mu \left[\partial_\mu A^\nu - \partial_\nu A^\mu \right] = - \left[\partial_\mu^2 A^\nu - \partial_\nu (\partial_\mu A_\mu) \right]$$

(*) becomes:

$A_\mu \to A_\mu + \partial_\mu a$

$$\boxed{\partial_\mu^2 A^\nu - \partial_\nu (\partial_\mu A_\mu) = J^\nu(x)}$$

$\partial_\mu^2 A^\nu +$

$+ \cancel{\partial_\mu^2 \partial_\nu a}$

$- \partial_\nu (\partial_\mu A_\mu)$

$- \cancel{\partial_\nu \partial_\mu^2 a} = 0$

$\Rightarrow$ For Leptonic Field

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \ldots} = \frac{\partial \mathcal{L}}{\partial \ldots}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} \qquad \frac{\partial \mathcal{L}}{\partial \psi}$$

$$0 =$$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \left(\gamma^\mu \left[i\partial_\mu - eA_\mu\right] - m\right)\psi(x) = 0$$

(*) becomes

$$\boxed{\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right) - m\right]\psi(x) = 0}$$

⟹ for antifermions:

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \psi_{,\mu}} - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

$$0 =$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = \bar{\psi}\left[\gamma^\mu\left(i\partial_\mu - eA_\mu\right)\right] - m = 0$$

$$\psi^\dagger \gamma_0 \left[\gamma^\mu \left(+i\partial_\mu - eA_\mu\right)\right] = 0$$

$$\boxed{\bar{\psi}\gamma^\mu = \gamma^{\mu T}\,\bar{\psi}}$$

$$\rightarrow \left[\gamma^{\mu T}\left(i\partial_\mu - eA_\mu\right) - m\right]\bar{\psi} = 0$$

L

$$\underbrace{C\gamma^{\mu T}C^{+}}_{-\gamma^{\mu}}C\left[(i\partial_\mu - eA_\mu) - \overset{c}{m}\right]\bar{\psi}^T = 0$$

$$\boxed{C\gamma^{\mu T}C^{+} = -\gamma^{\mu}}$$

$$-\gamma^\mu C(i\partial_\mu - eA_\mu) - Cm\bar{\psi}^T = 0$$

$$\left[-\gamma^\mu(i\partial_\mu - eA_\mu) - m\right]C\bar{\psi}^T$$

$$\boxed{C\bar{\psi}^T = \psi^c(x)}$$

$$\left[\gamma^\mu(i\partial_\mu - eA_\mu) + m\right]\psi_c = 0$$

$$\left[\gamma^\mu(-i\partial_\mu + eA_\mu) - m\right]\psi_c = 0$$

$$\hat{C} = i\alpha_2 = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$$

Solution of the above equations:

$$\psi(x) = \sum \frac{1}{\ } \ldots (kx)\left[C_k e^{-ikx} + C_k^+ e^{ikx}\right]$$

$$A_\mu(x) = \sum_{k,\nu} \frac{}{\sqrt{2\omega_k}} e_\mu(k) \nu \left[C_{k\nu} e \quad C^+_{k\nu} e \right]$$

$$\Psi(x) = \sum_{p,\lambda=+\frac{1}{2},-\frac{1}{2}} \frac{1}{\sqrt{2\varepsilon_p}} \left[a_{p,\lambda} u(p,\lambda) e^{-ipx} + b_p^+ u^c(p,\lambda) e^{ipx} \right]$$

$$\boxed{u^c = \hat{C} \bar{u}(p,\lambda)}$$

$\omega = \sqrt{k^2} = |k|$

$\varepsilon_p = \sqrt{p^2 + m^2}$

$\sum_p = \int \frac{d^3 p}{(2\pi)^3}$

Polarizations

$e_\mu(k,\nu) = g_{\mu\nu}$ $\perp$ K

$\nu = 1, 2$ $C_{k\nu}$ physical

$\nu = 0, 3$ Non physical

Should not contribute In physical processes

- Amplitudes $u(p,\lambda)$ $u^c(p,\lambda)$
Solution of Dirac Equation at $e=0$

- commonly used normalization $\bar{u} u = 2m$

P

$$u(p,\lambda) = \begin{pmatrix} \sqrt{\varepsilon_p + m}\, \chi(\lambda) \\ \sqrt{\varepsilon_p - m}\, \vec{\sigma}\vec{n}\, \chi(\lambda) \end{pmatrix}$$

n – can be chosen for arbitrary z – λ – spin projection or $n \longrightarrow n_p = \vec{p}/p$ — along χ – Helicity

$$\boxed{\chi(\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi(-\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

– Other spin ~~states~~ are obtained by rotation around $\hat{n}$ in θ angle

$$\chi_{\omega}^{(\lambda)} = e^{-\frac{i}{2}\vec{\sigma}\cdot\hat{\theta}}\chi(\lambda)$$

⟹ Four Momenta of The Field

$$P = \frac{\partial L}{\partial \dot{q}} \longrightarrow P_\mu = \frac{\partial \mathcal{L}}{\partial \varphi_{,\mu}}$$

$$E = \dot{q}\frac{\partial L}{\partial \dot{q}} - L$$

$$T_\mu^{\ \nu} = \varphi_{,\mu}\frac{\partial \mathcal{L}}{\partial \varphi_{,\nu}} - \delta_\mu^{\ \nu}\mathcal{L}$$

$$P_{0\nu} = \varphi_{,0}\frac{\partial \mathcal{L}}{\partial \varphi_\nu} \quad i\gamma_\mu \, \eta\bar{\psi}$$

$$\bar{u}\gamma^\mu u = 2p^\mu$$

$$\frac{-i\varepsilon_p}{\sqrt{2\varepsilon_p}} \quad a^+_{p,\lambda} \quad \overline{u(p,\lambda)}\frac{\gamma^\mu u}{\sqrt{2\varepsilon}}$$

$$P_\mu^\gamma = \sum_{k,\nu} \left(\frac{1}{2}\right) k_\mu \left(c_{k,\nu}^\dagger c_{k,\nu} + c_{k,\nu} c_{k,\nu}^\dagger \right)$$

$$P_\mu^e = \sum_{p,\lambda} p_\mu \left(a_{p,\lambda}^\dagger a_{p,\lambda} - b_{p,\lambda} b_{p,\lambda}^\dagger \right)$$

$$Q = \int J_0(x)\,dx = \sum_{p,\lambda} \left(a_{p,\lambda}^\dagger a_{p,\lambda} + b_{p,\lambda} b_{p,\lambda}^\dagger \right)$$

$$J_0 = j_0/e$$

1.3 Quantization: S-matrix

$$[c_{k,\lambda}, c_{k,\lambda}^\dagger] = \delta_{kk'}\delta_{\nu,\nu'}$$

$$[c_{k,\lambda}, c_{k,\lambda}] = 0$$

$$\{a_{p,\lambda}, a_{p,\lambda}^\dagger\} = \delta_{pp'}\delta_{\lambda,\lambda'}$$

$$\{a_{p,\lambda}\ a_{p,\lambda'}\} = 0$$

$$N_{k,\nu}^\gamma = c_{k,\nu}^\dagger c_{k,\nu} \qquad \Big| \quad c_{k\nu} c_{k\nu}^\dagger = 1 + N_{k,\nu}^\gamma$$

$$N^-_{k,\lambda} = a^+_{p,\lambda} a_{p,\lambda}$$

$$N^+_{p,\lambda} = b^+_{p,\lambda} b_{p,\lambda}$$

$$P^\gamma_\mu = \sum_{k,\nu} \frac{1}{2} k_\mu \left(N^\gamma_{k,\lambda} + N^\gamma_{k,\lambda} + 1 \right) = \sum k_\mu N_{k,\nu} + \sum_{k,\nu} \frac{1}{2} k_\mu$$

$$P^e_\mu = \sum_{p,\lambda} p_\mu \left(N^-_e + N^+_e - 1 \right) =$$

$$= \sum_{p,\lambda} p_\mu (N^-_e + N^+_e) - \sum_{p,\lambda} p_\mu$$

$$Q = \sum_{p,\lambda} \left(a^+_{p,\lambda} a_{p,\lambda} + b_{p,\lambda} b^+_{p,\lambda} \right) =$$

$$= \sum_{p,\lambda} \left(N^-_{p,\lambda} - N^{(+)}_{p,\lambda} \right) + \sum_{p,\lambda} 1$$

$$E^\gamma_0 = \left(P^\gamma_0 \right)_0 = \sum_{k,\lambda} \frac{1}{2} \omega_k$$

$$E_0^e = (P^e)_0 = -\sum_{p,\lambda} \varepsilon_p$$

$$P_0 = \sum_{p,\lambda} 1$$

Fock states

$$\Phi^{\gamma}_{1k,\nu} = c^{+}_{k,\nu}\Phi_0 \qquad \Phi^{e}_{1p,\lambda} = a^{+}_{p,\lambda}\Phi_0$$

$$\Phi^{e^+}_{1p,\lambda} = b^{+}_{p,\lambda}\Phi_0$$

$$c_{k,\nu}\Phi_0 = 0 \qquad a_{p,\lambda}\Phi_0 = 0; \; b_{p,\lambda}\Phi_0 = 0$$

$\nu = 1,2$

S-matrix of Interaction

$t \to -\infty \qquad \Phi_e$

$t \to +\infty \qquad \Psi_e$

$$\Psi_e = \sum_m \Phi_m S_{me}$$

S_{me} - Transition amplitude from initial state e final state m

$W_{e\to m} = |S_{me}|$

$\Rightarrow$ Formel

$$\Psi_L = \hat{S}\,\phi_L$$

Matrix

$$\boxed{\hat{S} = T e^{\left[i\int_{-\infty}^{+\infty} \mathcal{L}' d^4x\right]}} =$$

$$= T \exp\left[-ie \int_{-\infty}^{+\infty} \bar{\Psi}(x)\gamma_\mu \Psi(x) A_\mu(x)\, d^4x\right]$$

$$\boxed{S_{me} = \phi_m^* \hat{S} \phi_e}$$

$$\hat{S} = T \prod_j e^{-ie\sum_j \bar{\Psi}(x_j)\gamma_\mu \Psi(x_j) A_\mu(x_j) d^4x_j}$$

Time ordered

Perturbation theory

$$S = \sum_n \frac{(-ie)^n}{n!} \int\int\int \cdots T\left((\bar{\psi} \not{A} \psi)_1 (\bar{\psi} \not{A} \psi)_2 \cdots (\bar{\psi} \not{A} \psi)_n \right) d^4x_1\, d^4x_2 \cdots d^4x_n$$

N- product

left side a_k^{+} a_k right side

4.

(r)

$$d_{\mu\nu} = g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}$$ Feynman Gauge

$$D^0_{\mu\nu}(x) = \int e^{-ikx} D^0_{\mu\nu}(k) \frac{d^4k}{(2\pi)^4}$$

$$\left[g_{\mu\sigma}\partial_x^2 - \left(1-\frac{1}{\xi}\right)\partial_\mu \partial_\sigma\right] D_{\sigma\nu}(x) = g_{\mu\nu}\,\delta^4(x)$$