Lecture15

Sunday, November 26, 2017 11:30 PM

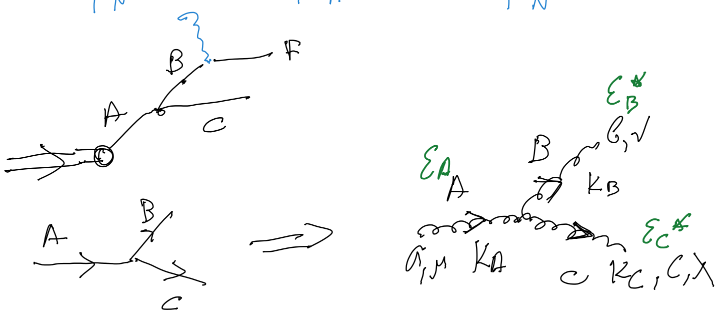
QCD Evolution Equation III

$$\tilde{f}(x) = \int_0^1 dy \int_0^1 dz\, \delta(zy - x)\, f(y) \left[\delta(z-1) + dP_{BA}(z)\right]$$

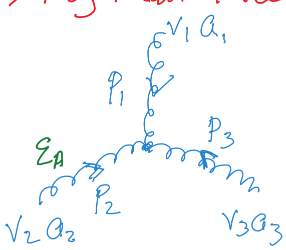
where

$$dP_{BA}(z)\, dz = \frac{1}{8\pi^2} \frac{z(1-z)}{2} \sum_{spin} \frac{|V_{A\to B+C}|^2}{P_\perp^2}\, dz\, dP_\perp^2 \qquad \text{Eq. 1}$$

$$y = \frac{K_A^z}{P_N} \qquad z = \frac{K_B^z}{K_A^z}, \qquad x = \frac{K_B^z}{P_N}$$



⟹ Feynman Rule of 3g interaction



$$= -g f^{a_1 a_2 a_3} \left[g^{\nu_1 \nu_2} (P_1 - P_2)^{\nu_3} + g^{\nu_2 \nu_3} (P_2 - P_3)^{\nu_1} + g^{\nu_3 \nu_1} (P_3 - P_1)^{\nu_2} \right]$$

$$V_{3g} = -g f^{abc} \left[g^{\mu\nu} (-K_B - K_A)^\lambda + g^{\lambda\mu} (K_A + K_C)^\nu + g^{\nu\lambda} (-K_C + K_B)^\mu \right]$$

$$-iM_{\lambda_a \lambda_b \lambda_c} = g f^{abc}_{\text{(fixed)}} \Big[\underbrace{(K_A + K_B)\varepsilon_C^+ (\varepsilon_A \varepsilon_B^+)}_{\text{I}} - \underbrace{(K_A + K_C)\varepsilon_B^+ (\varepsilon_A \varepsilon_C^+)}_{\text{II}} + \underbrace{(K_C - K_B)\varepsilon_A (\varepsilon_C^+ \varepsilon_B^+)}_{\text{III}} \Big] \qquad \text{Eq. 2}$$

⇒ Momenta of Gluons and Their wave function

For Generic Momentum

$K = (E, P_z, P_x, P_y)$ $\qquad \varepsilon_{\pm} = \mp\sqrt{\tfrac{1}{2}}\left(0, -\frac{P_x \pm iP_y}{P_z}, 1, \pm i\right)$

for $P_z \to \infty$

$K \cdot P = 0 \cdot E - \left[\left(P_x \pm iP_y\right) + \left(P_x \pm iP_y\right)\right] = 0$

$|\varepsilon_{\pm}|^2 = \frac{1}{2}\left(0 - \left(\frac{(P_x^2 + P_y^2)}{P_z^2} - 1 - 1\right)\right) = -1$

$\varepsilon_+ \varepsilon_-^+ = \frac{1}{2}\left(0 - \left(-\frac{P_x^2 + P_y^2}{P_z^2} - 1 + 1\right)\right) = 0$

For our Momenta

$K_A = (P, P, 0\,0)$ $\qquad \varepsilon_A^{\pm} = \mp\sqrt{\tfrac{1}{2}}\left(0, 0, 1, \pm i\right)$

$K_B = \left(zP + \frac{P_\perp^2}{2zP}, zP, P_\perp\right)$ $\qquad \varepsilon_B^{\pm} = \mp\frac{1}{\sqrt{2}}\left(0, -\frac{P_x \pm iP_y}{zP}, 1, \pm i\right)$

$K_C = \left((1-z)P + \frac{P_\perp^2}{2(1-z)P}, (1-z)P, -P_\perp\right)$ $\qquad \varepsilon_C^{\pm} = \mp\sqrt{\tfrac{1}{2}}\left(0, \frac{P_x \pm iP_y}{(1-z)P}, 1, \pm i\right)$

From Eq(2) we calculate

$$-iM_{\substack{+++\\ABC}} = gf^{abc}\Big[\left((K_A + K_B)\,\varepsilon_{C+}^{+}\right)\left(\varepsilon_{A+}\,\varepsilon_{B+}^{+}\right) - \left((K_A + K_C)\,\varepsilon_{B+}^{+}\right)\left(\varepsilon_{A+}\,\varepsilon_{C}^{+}\right) + \left((K_C - K_B)\,\varepsilon_{A+}\right)\left(\varepsilon_{C+}^{+}\,\varepsilon_{B+}^{+}\right)\Big]$$ Eq(3)

We need to Estimate

$\left(\varepsilon_{A+}\,\varepsilon_{B+}^{+}\right) = \frac{1}{2}\left(0 - 0 - 1 - (-i)(i)\right) = -1$

$(K_A + K_B)\,\varepsilon_{C+}^{+} = \frac{1}{\sqrt{2}}\left(0, -(K_A + K_B)^z\left(-\frac{P_x - iP_y}{(1-z)P}\right) - K_B^x(1) - K_B^y(i)\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{(P + zP)(P_x - iP_y)}{(1-z)P} + P_x - iP_y\right) =$

$= \frac{P_x - iP_y}{\sqrt{2}}\left(\frac{1+z}{1-z} + 1\right) = \frac{\sqrt{2}\,(P_x - iP_y)}{1-z}$

$$\varepsilon_{A+}\varepsilon^+_{C+} = \frac{1}{2}\left(0-0-1-(-i\,i)\right) = -1$$

$$(k_A+k_C)\,\varepsilon^+_{B+} = \frac{1}{\sqrt{2}}\left(0-(k_A+k_C)^z\frac{p_x-ip_y}{zp} - k_C^x(-1) - k_y\, i\right)$$

$$= \frac{1}{\sqrt{2}}\left(-(p+(1-z)p)\frac{p_x-ip_y}{zp} - p^x + ip^y\right) =$$

$$= \frac{1}{\sqrt{2}}(p_x-ip_y)\left[-\frac{(2-z)}{z} - 1\right]$$

$$= -\frac{1}{\sqrt{2}}(p_x-ip_y)\left[\frac{2-z+z}{z}\right] = -\sqrt{2}\,\frac{(p_x-ip_y)}{z}$$

$$\varepsilon^+_{C+}\varepsilon^+_{B+} = \frac{1}{2}\left(0+\frac{(p_x-ip_y)^2}{z(1-z)p^2} - 1 - (-i)^2\right) = 0$$

Inserting to Eq(3)

$$-iM_{+++} = g f^{abc}\left[-\frac{\sqrt{2}\,(p_x-ip_y)}{1-z} - \frac{\sqrt{2}\,(p_x-ip_y)}{z}\right] =$$

$$= -\sqrt{2}\,(p_x-ip_y)\, g f^{abc}\left(\frac{1}{1-z}+\frac{1}{z}\right) =$$

$$= -\sqrt{2}\,(p_x-ip_y)\, g f^{abc}\left(\frac{1}{(1-z)z}\right)$$

Now we calculate

$$-iM^{+-+}_{ABC} = g f^{abc}\Big[\big((k_A+k_B)\,\varepsilon^+_{C+}\big)(\varepsilon_{A+}\varepsilon^+_{B-}) -$$

By

$$-\big((k_A+k_C)\,\varepsilon^+_{B}\big)(\varepsilon_{A}\ \varepsilon^+_{\ldots})$$

$$((K_A+K_C)\,\varepsilon_{B-})\,(\varepsilon_{A+}\,\varepsilon_{C+})$$

$$+((K_C-K_B)\,\varepsilon_{A+})\,(\varepsilon_{C+}^{+}\,\varepsilon_{B-}^{+})\Big]$$

$$(\varepsilon_{A+}\,\varepsilon_{B-}^{+}) = \frac{1}{2}\Big(0-0-\underbrace{(-1)(1)}_{-1}-\underbrace{(-i)(i)}_{1}\Big) = 0$$

$$\varepsilon_{A+}\,\varepsilon_{C+}^{+} = -1$$

$$(K_A+K_C)\,\varepsilon_{B-}^{+} = \frac{1}{\sqrt{2}}\Big(0-(K_A+K_C)^{z}\Big(-\frac{P_x+iP_y}{zp}\Big)-K_C^{x}\,1-K_C^{y}\,i\Big)$$

$$=\frac{1}{\sqrt{2}}\Big((p+(1-z)p)\frac{P_x+iP_y}{zp}+(P_x+iP_y)\Big)=$$

$$=\frac{1}{\sqrt{2}}(P_x+iP_y)\Big(\frac{2-z}{z}+1\Big)=\frac{\sqrt{2}(P_x+iP_y)}{z}$$

$$\varepsilon_{C+}^{+}\,\varepsilon_{B-}^{+} = \frac{1}{2}\Big(0-\Big(-\frac{P_x-iP_y}{(1-z)p}\Big)\Big(-\frac{P_x+iP_y}{zp}\Big)-(-1)(1)-(i)(i)\Big)$$

$$=\frac{1}{2}\Big(-\frac{P_x^2+P_y^2}{z(1-z)p^2}+1+1\Big)=1$$

$$(K_C-K_B)\,\varepsilon_{A+} = \frac{1}{\sqrt{2}}\Big(0-0-(K_C^{x}-K_B^{x})(-1)-(K_C^{y}-K_B^{y})(-i)\Big)$$

$$=\frac{1}{\sqrt{2}}\Big((-2P_x)+i(-2P_y)\Big)=-\sqrt{2}\,(P_x+iP_y)$$

$$-iM_{+-+} = gf^{abc}\Big[\frac{\sqrt{2}(P_x+iP_y)}{z}-\sqrt{2}(P_x+iP_y)\Big]=$$

$$= gf^{abc}\sqrt{2}(P_x+iP_y)\Big[\frac{1}{z}-1\Big]=gf^{abc}\sqrt{2}(P_x+iP_y)\Big[\frac{1-z}{z}\Big]$$

Now we calculate

$$-i M_{\underset{ABC}{++-}} = g f^{abc}\Big[(k_A + k_B)\,\varepsilon_{C-}^{+} (\varepsilon_{A+}\varepsilon_{B+}^{+}) - (k_A + k_C)\,\varepsilon_{B+}^{+} (\varepsilon_{A+}\varepsilon_{C-}^{+}) + (k_C - k_B)\,\varepsilon_{A+} (\varepsilon_{C-}^{+}\varepsilon_{B+}^{+})\Big]$$

$$\varepsilon_{A+}\varepsilon_{B+}^{+} = -1$$

$$(k_A + k_B)\varepsilon_{C-}^{+} = \left(0 - (k_A^z + k_B^z)\left(\frac{1}{\sqrt{2}}\frac{P_x + iP_y}{(1-z)P}\right) - \frac{k_B^x - k_B^y i}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left[\frac{(P + zP)}{(1-z)P}(P_x + iP_y) - P_x + iP_y\right] = -\frac{P_x + iP_y}{\sqrt{2}}\left(\frac{1+z}{(1-z)} + 1\right) =$$

$$= -\frac{\sqrt{2}(P_x + iP_y)}{1-z}$$

$$\varepsilon_{A+}\varepsilon_{C-}^{+} = \frac{1}{2}\left(0 - 0 - (-1)(1) - (-i)(i)\right) = \frac{1}{2}(1 - 1) = 0$$

$$(\varepsilon_{C-}^{+}\varepsilon_{B}^{+}) = \frac{1}{2}\left(0 - \frac{P_x + iP_y}{(1-z)P}\frac{(P_x - iP_y)}{zP} - \underbrace{(1)(-1)}_{1} - \underbrace{(i)(i)}_{1}\right) = 1$$

$$(k_C - k_B)\varepsilon_{A+} = \frac{1}{\sqrt{2}}\left(0 - 0 - (k_C - k_B)^x(-1) - (k_C - k_B)^y(-i)\right) =$$

$$= \frac{1}{\sqrt{2}}\left(-2P_x - 2P_y i\right) = -\sqrt{2}(P_x + iP_y)$$

$$-i M_{++-} = g f^{abc}\left[-\frac{\sqrt{2}(P_x + iP_y)}{1-z}(-1) - \sqrt{2}(P_x + iP_y)(1)\right] =$$

$$= g f^{abc}\sqrt{2}(P_x + iP_y)\left[\frac{1}{1-z} - 1\right] = g f^{abc}\sqrt{2}(P_x + iP_y)\left[\frac{z}{1-z}\right]$$

$$-\partial\, \partial' \; J_2(P_x + i P_3) \Big[\frac{\quad}{1-z} \quad \Big] \qquad \frac{\quad}{(1-z)}$$

Consider

$$-i M_{\substack{+--\\ABC}} = g f^{abc} \Big[(K_A + K_B)\, \varepsilon^{+}_{C-} (\varepsilon_{A+} \varepsilon^{+}_{B-}) - (K_A + K_C)\, \varepsilon^{+}_{B-} (\varepsilon_{A+} \varepsilon^{+}_{C-}) + (K_C - K_B)\, \varepsilon_{A+} (\varepsilon^{+}_{C-} \varepsilon^{+}_{B-}) \Big]$$

$$\varepsilon_{A+} \varepsilon^{+}_{B-} = 0$$

$$\varepsilon_{A+} \varepsilon^{+}_{C-} = 0$$

$$\varepsilon^{+}_{C-} \varepsilon^{+}_{B-} = \frac{1}{2}\left(0 - \frac{(P_x + i P_3)}{(1-z)P} \frac{(-P_x + i P_3)}{zP} - 1 - (i)(i) \right) = 0$$

$$-i M_{\substack{---\\ABC}} = 0$$

$$|M_{+++}|^2 + |M_{+-+}|^2 + |M_{++-}|^2 =$$

$$= 2 (P_x^2 + P_3^2)\, g^2 f^{abc} f^{abs} \times \Big[$$

$$\times \left[\frac{1}{(1-z)^2 z^2} + \frac{(1-z)^2}{z^2} + \frac{z^2}{(1-z)^2} \right]$$

(a)

$$ⓐ = \left[\frac{1+(1-z)^4+z^4}{(1-z)^2 z^2}\right] = \left[\frac{1+(1-2z+z^2)^2+z^4}{(1-z)^2 z^2}\right]$$

$$= \left[\frac{1+(1-2z)^2+2z^2(1-2z)+z^4+z^4}{(1-z)^2 z^2}\right] =$$

$$= \left[\frac{1+1-4z+4z^2+2z^2-4z^3+2z^4}{(1-z)^2 z^2}\right] =$$

$$= \left[\frac{2-4z+6z^2-4z^3+2z^4}{z^2(1-z)^2}\right] =$$

$$= 2\left[\frac{1-2z+3z^2-2z^3+z^4}{z^2(1-z)^2}\right] =$$

$$= 2\left[\frac{(1-z)^2+z^2+\overbrace{z^2-2z^3+z^4}^{z^2(1-2z+z^2)}}{z^2(1-z)^2}\right] =$$

$$= 2\left[\frac{(1-z)^2+z^2+z^2(1-z)^2}{z^2(1-z)^2}\right] =$$

$$= 2\left[\frac{(1-z)}{\ldots} \ldots z \ldots z^2(1-z)^2 \ldots\right]$$

$$\frac{}{z(1-z)}\left[\frac{(1-z)}{z}+\frac{}{(1-z)}+z(1-z)\right]$$

$$\frac{1}{2}\frac{1}{N^2-1}|M|^2 = \frac{1}{2}\frac{g^2 f^{abc}f^{abc}}{N^2-1}\frac{2P_\perp^2\cdot 2}{z(1-z)}\left[\frac{1-z}{z}+\frac{z}{(1-z)}+z(1-z)\right]$$

$C_2(R)$

(*) $\frac{1}{2}$ - here is averaging by the spin of the gluon, it should be applied if we include also $|M_{---}|^2+|M_{-+-}|^2+|M_{--+}|^2$

Remember

but we used $M_{+++}=M_{---}$, $M_{++-}=M_{-+-}$, $M_{+-+}=M_{-+-}$ and did not averaged by spin

$$dP_{GG} = \frac{1}{8\pi^2}\frac{z(1-z)}{2}\frac{1}{P_\perp^2}\left[\frac{g^2 C_2(R)4P_\perp^2}{z(1-z)}\left[\frac{1-z}{z}+\frac{z}{(1-z)}+z(1-z)\right]\right]\times d\ln P_\perp^2$$

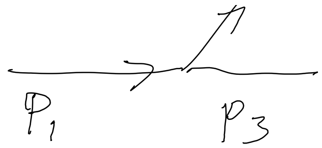
$$= \frac{g^2}{8\pi^2}2C_2(G)\left[\frac{1-z}{z}+\frac{z}{(1-z)}+z(1-z)\right]d\ln P_\perp^2$$

$$= \frac{\alpha}{2\pi}2C_2(G)\left[\frac{1-z}{z}+\frac{z}{(1-z)}+z(1-z)\right]d\ln P_\perp^2 =$$

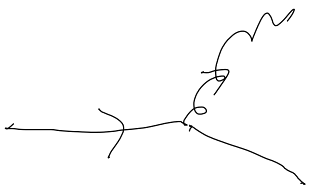
$$= \frac{\alpha}{2\pi}P_{GG}(z)\,d\ln P_\perp^2$$

Summarizing

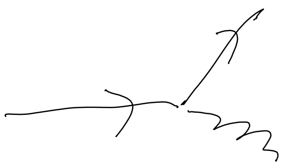
$$dP_{..} = \frac{\alpha}{}P_{DD}(z)\,d\ln P_\perp^2$$



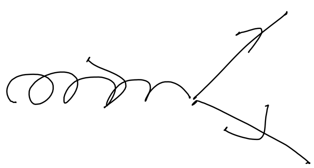
$d\Gamma_{P_1 P_2} = \frac{\ }{2\pi} \, P_{1,2} \ldots$



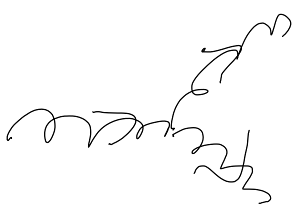
$$P_{Gq}(z) = C_2(R)\left[\frac{1+(1-z)^2}{z}\right]$$



$$P_{qq}(z) = C_2(R)\left[\frac{1+z^2}{1-z}\right]$$



$$P_{qG}(z) = \frac{1}{2}\left(z^2+(1-z)^2\right)$$



$$P_{GG}(z) = 2C_2(G)\left[\frac{1-z}{z}+\frac{z}{1-z}+z(z-1)\right]$$

$$C_2(R) = \frac{N^2-1}{2N} \qquad C_2(G) = \frac{1}{N^2-1}\sum_{abc} f^{abc} f^{abc} = N$$

$$\sum_d \sum_c \sum_{ab} f^{abd} f^{abc} = N\delta^{cd} = N(N^2-1)$$

$\Rightarrow$ So far we considered $z<1$

Remembering, Now,

$$\tilde{f}(x, p_{\perp max}^2, p_{\mu}^2) = \int_0^1 dy \int_0^1 dz\, \delta(zy - x) f(y) \left[\delta(z-1) + dP_{BA}(z)\right] =$$

$$= \int_0^1 dy \int_0^1 dz\, \delta(zy - x) f(y)\, \delta(z-1)$$

$$+ \int_0^1 dy \int_0^1 dz\, \delta(zy - x) \int_{p_{\perp min}^2}^{p_{\perp max}^2} f(y, p_\perp) \frac{\alpha(p_\perp)}{(2\pi)} P_{BA}(z)\, d \ln p_\perp^2$$

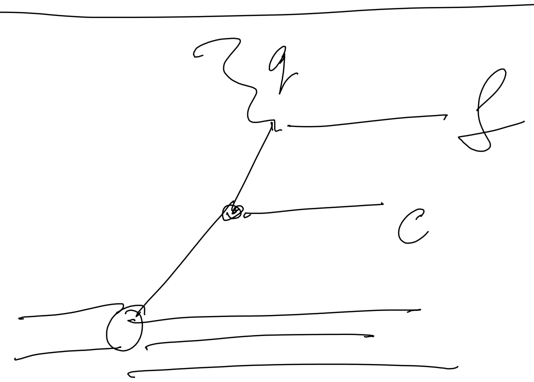
$E_g(\ldots)$

$$\boxed{p_{\perp min}^2 = \mu^2}$$ — μ^2 – characteristic scale

=> to calculate $p_\perp^{max}$ – limit

we consider the kinematics

of the DIS Reaction



$(P_+, 0, 0\ 0)$

$P_N = (P, P, 0\ 0)$

$$x = \frac{Q^2}{2Pq} = \frac{Q^2}{2m q_0^{Lab}}$$

$$P_N q_N = m q_0^{Lab}$$

— we have to choose q^μ — to satisfy above conditions as well as $q^2 = -Q^2$

$$q = \left(\frac{m q_0}{2P}, -\frac{m q_0}{2P}, +\sqrt{Q^2} \right) \quad (*)$$

① $q^2 = \left(\frac{m q_0}{2P}\right)^2 - \left(\frac{m q_0}{2P}\right)^2 - Q^2 = -Q^2$

② $P_N q = \frac{m q_0}{2} + \frac{m q_0}{2} = m q_0$

From (*) follows that max transverse momentum is Q^2

Therefore for Eq (**) one obtains

$$\tilde{f}(x, \mu^2, Q^2) = \int_0^1 dy \int_0^1 dz\, \delta(zy - x) f(x) \delta(z-1)$$

$$+ \int_0^1 dy \int_0^1 dz\, \delta(zy - x) \int^{\ln Q^2} f(y, P_\perp) \frac{\alpha(P_\perp)}{(2\pi)} P_{BA}(z) d\ln P_\perp$$

$\ln \mu^2$

⇒ from above we obtain

$$\frac{d f(x,\mu^2,Q^2)}{d \ln Q^2} = \frac{\alpha(Q^2)}{2\pi} \int_x^1 \frac{dy}{y} f(y,\mu^2) P_{BA}\left(\frac{x}{y}\right)$$

or

$$\frac{d f(x,t)}{dt} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} f(y,t) P_{BA}\left(\frac{x}{y}\right)$$

$$t = \ln Q^2$$