Lecture16

Monday, December 4, 2017 2:56 PM

Evolution Equation IV

In the previous Lecture we obtained

$$\frac{df(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{dy}{y} f(y,t)\, P_{BA}\left(\frac{x}{y}\right)$$

Using now specific splitting functions we can write

$\Rightarrow$ for quark-parton distributions we use $q^i(x,t)$, i – flavor $u\ d\ s$, $c\ b\ t$, $\bar{u}\ \bar{d}\ \bar{s}$, $\bar{c}\ \bar{b}\ \bar{t}$

for Gluon-partonic distribution $G(x,t)$

$$\frac{dq^i(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[\sum_{j=1}^{2f} q^j(y,t)\, P_{q^i q^j}\left(\frac{x}{y}\right) + G(y,t)\, P_{q^i G}\left(\frac{x}{y}\right)\right]$$

$$\frac{dG(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[\sum_{j=1}^{2f} q^j(y,t)\, P_{G q^j}\left(\frac{x}{y}\right) + G(y,t)\, P_{GG}\left(\frac{x}{y}\right)\right]$$

i, j – quark, antiquark

In the considered leading approximation

① $P_{q^j q^i} = \delta_{ij} P_{qq}$

② if neglect quark masses

$P_{G q^i} = P_{Gq}$ (independent of i-flavor)

③ $P_{q^i G} = P_{qG} = P_{\bar{q}G}$ (independent of i)

With this conditions for above equations we obtain

$$\frac{d q^i(x)}{dt} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[q^i(y,t) P_{qq}\left(\frac{x}{y}\right) + G(y,t) P_{qG}\left(\frac{x}{y}\right) \right]$$

$$\frac{d G(x)}{dt} = \frac{\alpha(t)}{2\pi} \int_x^1 \frac{dy}{y} \left[\sum_{i=1}^{2f} q^i(y,t) P_{Gq}\left(\frac{x}{y}\right) + G(y,t) P_{GG}\left(\frac{x}{y}\right) \right]$$

– Some interpretation

$\frac{\alpha(t)}{2\pi} P_{Gq}(z)$ – probability density of finding G in q per unit t

P_{Gq} — prob. density with z momentum of fraction

same for

$$\frac{\alpha(t)}{2\pi} P_{qG}(z)$$

P_{qq} — prob density

– For Diagonal terms P_{qq} and P_{GG} is little bit different

we recall that,

$$\tilde{f}(x,t) = \int_0^1 dy \int_0^1 dz\, \delta(zy - x)\, q_i(y)\, \delta(z-1) +$$

$$+ \int_0^1 dy \int_0^1 dz\, \delta(zy - x) \int q(y, P_y) \frac{\alpha}{2\pi} P_{qq}(z)\, dt$$

Eq (4)

for dt units

$$+ \int_0^1 dy \int_0^1 dz\, \delta(zy - x) \int_{\text{lower}}^{\text{upper}} G(y, P_y) \frac{\alpha}{2\pi} P_{qG}(z)\, dt$$

$$= \int_x^1 \frac{dy}{y} q_i(y) \left[\delta\left(\frac{x}{y} - 1\right) + \frac{\alpha}{2\pi} P_{qq}\left(\frac{x}{y}, t\right) \right] + \int_x^1 \frac{dy}{y} G(y, P_y) \frac{\alpha}{2\pi} P_{qG}(t)$$

$$P(z, t, t+\delta t) = \delta(z-1) + \frac{\alpha}{2\pi} P_{qq}(z)$$

single $\quad P_{Gq}(z, t, t+\delta t) = \delta(z-1) + \frac{\alpha}{2\pi} P_{Gq}(z)\, dt$

Source case

⇒ Now consider

$$\int P_{q,q}(z,t,t+\delta t)\,dq$$ — total probability of finding quark in the quark

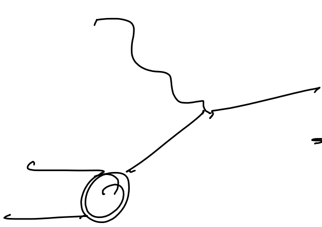
- since total number of quark-antiquark is conserved

$$\int P_{q,q}(z,t,t+\delta t)\,dq = 1$$

therefore $$\boxed{\int_0^1 dz\, P_{qq}(z) = 0}$$

- This is not the case for P_{GG} since total number of Gluons is not conserved

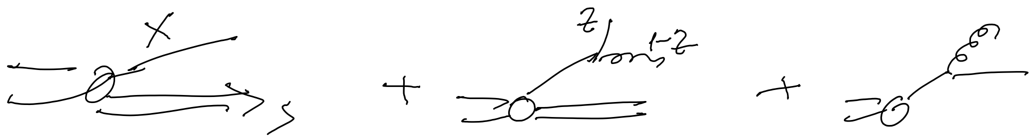
- Using the fact that total momentum of the Nucleon is not changed



$$f_i = P_N = \int x\left[\sum_i^{2f} q_i(x) + G(x)\right] dx = 1$$

① start from the situation in which

Nucleon originated one quark which splitted in dt



Splitting should not add extra momentum to the parent parton

– multiply Eq (4) by x and integrating for dt units

total momentum of nucleons

$$\frac{d}{dt}\int_0^1 dx\, x\left[\sum_{i=1}^{2f} q^i(x,t)+G(x,t)\right]=0$$

using Eq 4

$$\frac{d}{dt}\Bigg[\int_0^1 dx\cdot x\int_0^1 dy\int_0^1 dz\,\delta(zy-x)\Bigg[\sum_{i=1}^{2f} q_i(y)P_{qq}(z)$$

$$+\ 2f\,G(y,t)P_{qG}(z)\,dt$$

$$+\sum_{i=1}^{2f} q_i(y,t)P_{Gq}(z)\,dt+G(y,t)P_{GG}(z)\,dt\Bigg]$$

$$=\int_0^1 dy\cdot y\int_0^1 dz\cdot z\left(P_{qq}(z)+P_{Gq}(z)\right)q_i(y,t)$$

$$\int_0^1 dy\, y \int_0^1 dz\, z \left(2f\, P_{qG}(z) + P_{GG}(z)\right) G(y,t) = 0$$

$\Rightarrow$ Follows ~~that~~ that

$$\int_0^1 dz\, z \left[P_{qq}(z) + P_{Gq}(z)\right] = 0$$

$$\int_0^1 dz\, z \left[2f\, P_{qG}(z) + P_{GG}(z)\right] = 0$$

Summarizing the obtained conditions

$$P_{qq}(z) = P_{Gq}(1-z)$$

$$P_{qG}(z) = P_{qG}(1-z) \qquad (z<1)$$

$$P_{GG}(z) = P_{GG}(1-z)$$

$$\int_0^1 dz\, P_{qq}(z) = 0$$

$$\int^1 dz\, z \left[P_{qq}(z) + P_{Gq}(z)\right] = 0$$

$$\int_0^1 dz\, z\left[2f\, P_{qG}(z) + P_{GG}(z)\right] = 0$$

$$\left.\begin{aligned}
P_{qq}(z) &= C_2(R)\left[\frac{1+z^2}{1-z}\right] \\
P_{Gq}(z) &= C_2(R)\left[\frac{1+(1-z)^2}{z}\right] \qquad z<1 \\
P_{qG}(z) &= \frac{1}{2}\left(z^2+(1-z)^2\right) \\
P_{GG}(z) &= 2C_2(G)\left[\frac{1-z}{z}+\frac{z}{1-z}+z(1-z)\right]
\end{aligned}\right\}$$

Regularization

$$\frac{1}{1-z} \longrightarrow \frac{1}{(1-z)_+}$$

$$\int_0^1 \frac{dz\, f(z)}{(1-z)} \longrightarrow \int_0^1 \frac{dz\, f(z)}{(1-z)_+} = \int_0^1 dz\, \frac{f(z)-f(1)}{1-z} = -\int_0^1 d\ln(1-z)\,\big(f(z)-f(1)\big)$$

$$= -\ln(1-z)\,\big(f(z)-f(1)\big)\Big|_0^1 + \int_0^1 dz\, \ln(1-z)\,\frac{d}{dz}f(z)$$

Example

$$\int^1 dz\, \frac{1}{(1-z)} = 0$$

$z \qquad (1-z)_+$

Effect of all this

$\Rightarrow$ Consider $P_{qq}(z)$ $\qquad C_2(R) = \frac{8}{6} = \frac{4}{3}$

$$P_{qq}(z) = C_2(R)\left[\frac{(1+z)^2}{1-z}\right] + A\,\delta(1-z)$$

$$\int_0^1 dz\, P_{qq}(z) = 0$$

$$C_2(R)\int_0^1 dz\, \frac{1+z^2}{(1-z)_+} + A = 0$$

$$C_2(R)\int_0^1 \frac{z^2}{(1-z)_+}\,dz = C_2(R)\int_0^1 \frac{z^2-1}{(1-z)}\,dz = -C_2(R)\int_0^1 (1+z)\,dz$$

$$= -C_2(R)\left(z+\frac{z^2}{2}\right)\Big|_0^1 = -C_2(R)\left(1+\frac{1}{2}\right) = -C_2(R)\left[\frac{3}{2}\right]$$

$$A = \frac{3}{2}C_2(R)$$

$$P_{qq}(z) = C_2(R)\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right]$$

$\Rightarrow$ next we need to show

$$P_{GG}(z) = 2C_2(G)\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) + \left(\frac{11}{12} - \frac{1}{3}\frac{T(R)}{C_2(G)}\right)\delta(1-z)\right]$$

$$T(R) = \frac{1}{2}f$$