Lecture17

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Evolution Equation V

- From the Previous Lecture

$$\frac{dq^i(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[q^i(y,t)P_{qq}\left(\frac{x}{y}\right) + G(y,t)P_{qG}\left(\frac{x}{y}\right)\right]$$

$$\frac{dG(x)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[\sum_{i=1}^{2f} q^i(y,t)P_{Gq}\left(\frac{x}{y}\right) + G(y,t)P_{GG}\left(\frac{x}{y}\right)\right]$$

We Calculated Splitting Functions as

$$P_{qq}(z) = C_2(R)\left[\frac{1+z^2}{1-z}\right]$$

$$P_{Gq}(z) = C_2(R)\left[\frac{1+(1-z)^2}{z}\right] \qquad z<1$$

$$P_{qG}(z) = \frac{1}{2}\left(z^2+(1-z)^2\right)$$

$$P_{GG}(z) = 2\,C_2(G)\left(\frac{1-z}{z} + \frac{z}{1-z} + z(1-z)\right)$$

Obtained Conditions of Splitting Functions

$$\int_0^1 dz\, P_{qq}(z) = 0$$

$$\int_x^1 dz\cdot z\left[P_{qq}(z) + P_{Gq}(z)\right] = 0$$

$$\int_0^1 dz\, z \left[2f\, P_{qG}(z) + P_{GG}(z)\right] = 0$$

— Regularization of $\frac{1}{1-z}$ singularity

$\frac{1}{1-z} \to \frac{1}{(1-z)_+}$ such that

$$\int_0^1 dz \frac{f(z)}{(1-z)} \to \int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{1-z} =$$

$$= -\int_0^1 d\ln(1-z)\left(f(z)-f(1)\right) =$$

$$= \int_0^1 dz \ln(1-z) \frac{d}{dz} f(z)$$

$$\int_0^1 dz \frac{1}{(1-z)_+} = 0$$

$\Rightarrow$ Using Above Regularization we obtained

$$P_{qq}(z) = C_2(R)\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(z-1)\right] \text{ for all } z$$

$\Rightarrow$ Now we consider P_{GG}

$$P_{GG}(z) = 2C_2(G)\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\right] + B\delta(1-z)$$

$$\int^1 dz\, z \left[2f\, P_{qG}(z) + P_{GG}(z)\right] = 0 =$$

$$\int_0 dz\, z\, [\ldots + 14 G(z) + \ldots G(z)]$$

$$= \int_0^1 dz \cdot z \Big[2f\Big(\frac{1}{2}\big(z^2 + (1-z)^2\big)\Big) +$$

$$+ 2C_2(G)\Big(\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z)\Big)\Big]$$

$$+ B\int_0^1 dz \cdot z\, \delta(1-z) \cdot =$$

$$= f \int_0^1 dz \underbrace{\big(z^3 + z(1-z)^2\big)}_{a} +$$

$$+ 2C_2(G)\left[\int_0^1 dz\Big[(1-z) + \frac{z^2}{(1-z)_+} + z^2(1-z)\Big]\right] + B = 0$$

b

$$a = \int_0^1 dz\,(z^3 + z - 2z^2 + z^3) = \int_0^1 dz\,(2z^3 + z - 2z^2) =$$

$$= \Big|_0^1 \Big(\frac{2}{4}z^4 + \frac{z^2}{2} - \frac{2}{3}z^3\Big) = \frac{2}{4} + \frac{1}{2} - \frac{2}{3} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$b = \int_0^1 dz\Big[1 - z + z^2 - z^3 + \frac{\overset{(z-1)(z+1)}{z^2 - 1}}{1-z}\Big] = \int_0^1 dz\,[1 - z + z^2 - z^3 - z - 1]$$

$$= \int_0^1 dz\,[-z^3 - 2z + z^2] = -\frac{1}{4} - 1 + \frac{1}{3} = -\frac{5}{4} + \frac{1}{3} = \frac{-15+4}{12} = -\frac{11}{12}$$

$$\frac{f}{3} - 2C_2(G)\frac{11}{12} + B = 0$$

$$B = 2C_2(G)\frac{11}{12} - \frac{f}{3} = 2C_2(G)\Big[\frac{11}{12} - \frac{f}{2C_2(G)}\frac{1}{3}\Big]$$

$$P_{GG}(z) = 2C_2(G)\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + \left(\frac{11}{12} - \frac{1}{3}\frac{T(R)}{C_2(G)}\right)\delta(z-1)\right]$$

$$T(R) = \frac{f}{2}, \quad C_2(G) = N, \quad C_2(R) = \frac{N^2-1}{2N}$$

Final Form of Evolution Equation

$$\frac{dq^i(x,t)}{dt} = \frac{\alpha}{2\pi}\int_x^1 \frac{dy}{y}\left[q^i(y,t)\,P_{qq}\left(\frac{x}{y}\right) + G(y,t)\,P_{qG}\left(\frac{x}{y}\right)\right]$$

$$\frac{x}{y} = z \qquad dz = \frac{x}{y^2}\,dy \qquad \frac{dy}{y} = y\frac{dz}{x}$$

$$y = \frac{x}{z} \qquad dz = \frac{z\,dy}{y} \Rightarrow \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dq(x,t)}{dt} = \frac{\alpha}{2\pi}\int_x^1 \frac{dz}{z}\left(q_i\left(\frac{x}{z}\right)P_{qq}(z) + G\left(\frac{x}{z}\right)P_{qG}(z)\right)$$

$$P_{qq}(z) = C_2(R)\left[\frac{(1+z^2)}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

$$P_{qG}(\cancel{z}) = \frac{1}{2}\left((1-\cancel{z})^2 + \cancel{z}^2\right)$$

$\Rightarrow$ Consider the P_{qq} part

$$\frac{\alpha}{2\pi}\int_x^1 \frac{dz}{z}\left(q_i\left(\frac{x}{z}\right)\right) C_2\left[\frac{(1+z^2)}{(1-\cancel{z})_+} + \frac{3}{2}\delta(1-z)\right] =$$

$$= \frac{\alpha}{2\pi}\int_x^1 \frac{dz}{z}\left(q_i\left(\frac{x}{z}\right)\right) C_2\left[\frac{(1+z^2)}{(1-z)_+}\right] +$$

$$\frac{\alpha}{2\pi} C_2\, q_i(x)\cdot\frac{3}{2} + \frac{\alpha C_2}{2\pi}\int_x^1 \frac{dz}{z}\, q_i\left(\frac{x}{z}\right)\frac{(1+z^2)}{(1-z)_+}$$

$$\int_x^1 \frac{dz}{z}\, \boxed{q_i\left(\frac{x}{z}\right)\frac{(1+z^2)}{(1-z)_+}} = f(z)$$

f

Eq (Reg)

using

$$\int_x^1 dz \frac{f(z)}{(1-z)_+} = \int_{\cancel{x}}^1 dz \frac{(f(z) - f(1))}{1-z} - f(1)\int_x dz \frac{1}{1-z}$$

$$\int_x^1 dz\, \frac{q_i\left(\frac{x}{z}\right)(1+z^2)/z}{(1-z)_+} = \int_0^1 \frac{dz}{1-z}\left[q\left(\frac{x}{z}\right)\frac{(1+z^2)}{z} - 2q(x)\right]$$

$$- 2q_i(x)\int_0^x dz\, \frac{1}{1-z} =$$

$$= \int_0^1 \frac{dz}{1-z}\left(\frac{1+z^2}{z}q_i\left(\frac{x}{z}\right) - 2q_i(\cancel{x})\right) + 2q_i(x)\ln(1-x)$$

P_{qq} Part becomes

$$\frac{\alpha}{2\pi} C_2 \cdot \frac{3}{2}\, q_i(x, Q^2) + \frac{\alpha}{2\pi} C_2 \cdot 2 q_i(x) \ln(1-x)$$

$$+ \frac{\alpha}{2\pi} C_2 \int_x^1 \frac{dz}{1-z}\left(\frac{1+z^2}{z} q_i\left(\frac{x}{z}, Q^2\right) - 2q_i(x, Q^2)\right)$$

$$= \frac{\alpha}{2\pi} C_2(R) \cdot \frac{3}{2}\left[1 + \frac{4}{3}\ln(1-x)\right] q_i(x, Q^2)$$

$$+ \frac{\alpha}{2\pi} C_2(R) \int_x^1 \frac{dz}{1-z}\left(\frac{1+z^2}{z} q_i\left(\frac{x}{z}, Q^2\right) - 2q_i(x, Q^2)\right)$$

→ The Masles – Boltzmann Equation

will be for quark distribution

$$\frac{d q_i(x,Q^2)}{dt} = \frac{\alpha}{2\pi}\left\{\frac{3}{2}C_2(R)\left[1+\frac{4}{3}\ln(1-x)\right]q_i(x,Q^2)\right.$$
$$+\,C_2(R)\int_x^1 \frac{dz}{1-z}\left(\frac{1+z^2}{z}\,q_i\left(\frac{x}{z},Q^2\right)-2q_i(x,Q^2)\right)$$
$$+\int_x^1 dz\,\frac{(1-z)^2+z^2}{2z}\,G\left(\frac{x}{z},Q^2\right)$$

How Eq (Reg) is obtained.

$$\int_x^1 dz\,\frac{f(z)}{(1-z)_+} = \int_0^1 dz\,\frac{f(z)}{(1-z)} - \int_0^x dz\,\frac{f(z)}{(1-z)}$$

Regularization

$$= \int_0^1 dz\,\frac{(f(z)-f(1))}{(1-z)} - \int_0^x dz\,\frac{f(z)}{(1-z)} =$$

$$= \int^1 \frac{dz\,(f(z)-f(1))}{\ldots} + \int^x \frac{dz\,(f(z)-f(1))}{\ldots} - \int^x \frac{dz\,f(z)}{\ldots} =$$

$$\int_x^1 \quad 1-t \qquad \int_0 (1-z) \qquad \int_0 (1-z)$$

$$= \boxed{\int_x^1 \frac{dz\,(f(z)-f(1))}{1-z} - f(1)\int_0^x \frac{dz}{1-z}}$$

$\Rightarrow$ Can be Generalized

$$\boxed{\int_x^1 dz\, \frac{f(z)\, g(z)}{(1-z)_+} = \int_x^1 \frac{dz\,(f(z)-f(1))\,g(z)}{1-z} - f(1)\int_0^x \frac{dz\, g(z)}{1-z}}$$

$\Rightarrow$ In a similar manner we can obtain the Evolution Equation for Gluon distribution

$$\frac{d\,G(x,t)}{dt} = \frac{\alpha(t)}{2\pi}\int_x^1 \frac{dy}{y}\left[\sum_{i=1}^{2f} q^i(y,t)\,P_{Gq}\left(\frac{x}{y}\right) + G(y,t)\,P_{GG}\left(\frac{x}{y}\right)\right] = //$$

$\frac{x}{y} = z \qquad \frac{dy}{y} = \frac{dz}{z}, \qquad y = \frac{x}{z}$

$$// = \frac{\alpha}{2\pi}\int^1 dz \left[\sum^{2f} q^i(x, \ldots)\,P_{Gq}(z) + G\left(\frac{x}{z}, t\right) P_{GG}(z)\right] = \ldots$$

$$\left(1-\frac{\alpha}{2\pi}\right)\int_x \frac{dz}{z}\left[\sum_{i=1} q^i\left(\frac{x}{z}\right)\ldots G\left(\frac{x}{z}\right)\ldots\right]=$$

$$= \frac{\alpha}{2\pi}\left\{\int_x^1 \frac{dz}{z}\left[\sum_{i=1}^{2f} q^i\left(\frac{x}{z}\right) C_2(R)\left[\frac{1+(1-z)^2}{z}\right] + \underbrace{G\left(\frac{x}{z}\right) 2C_2(G) \times \left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + B_0\,\delta(1-z)\right]}_{6}\right\}$$

$$\boxed{B_0 = \left(\frac{11}{12} - \frac{1}{3}\frac{T(R)}{C_2(G)}\right)}$$

$E_q(G)$

6 $$\int_x^1 \frac{dz}{z} G\left(\frac{x}{z}\right) 2C_2(G)\left[\frac{1-z}{z} + \frac{z}{(1-z)_+} + z(1-z) + B_0\,\delta(1-z)\right]$$

$$= \int_x^1 \frac{dz}{z} G\left(\frac{x}{z}\right) 2C_2(G)\left[\frac{1-z}{z} + z(1-z)\right] + 2C_2(G)\underbrace{\int_x^1 \frac{dz}{z} G\left(\frac{x}{z}\right)\frac{z}{(1-z)_+}}_{62}$$

$$+ G(x)\, 2C_2(G)\cdot B_0$$

62 $$\int_x^1 dz\, G\left(\frac{x}{z}\right)\frac{1}{(1-z)_+} = \int_x^1 dz \frac{\left(G\left(\frac{x}{z}\right) - G(x)\right)}{1-z} - \int_0^x dz \frac{G(x)}{1-z}$$

$$= \int_x^1 dz \frac{\left(G\left(\frac{x}{z}\right) - G(x)\right)}{1-z} + G(x)\ln(1-x)$$

6 $$2C_2(G)\left[\int_x^1 \frac{dz}{z} G\left(\frac{x}{z}\right)\left[\frac{1-z}{z} + z(1-z)\right] + \int_x^1 dz \frac{G\left(\frac{x}{z}\right) - G(x)}{1-z}\right.$$

$$\left. \ldots G(x)\ln(1-x) + G(x)\cdot B_0\right] =$$

$+\ G(x)\ln(1-x)\ \ldots\ ]$

$$2C_2(G)\left[\int_x^1 \frac{dz}{z} G\left(\frac{x}{z}\right)\left[\frac{1-z}{z}+z(1-z)\right]+\left(B_0+\ln(1-x)\right)G(x)\right]$$

$$+\int_x^1 dz\,\frac{G\left(\frac{x}{z}\right)-G(x)}{1-z}$$

$\Rightarrow$ Including this in Eq (G)

$$\boxed{\frac{dG(x,t)}{dt}=\frac{\alpha}{2\pi}\left\{C_2(R)\int_x^1\frac{dz}{z}\left[\sum_{i=1}^{2n_f} q^i\left(\frac{x}{z},t\right)\right]\left[\frac{1+(1-z)^2}{z}\right]+2C_2(G)\left[\int_x^1\frac{dz}{z}G\left(\frac{x}{z},t\right)\left[\frac{1-z}{z}+z(1-z)\right]+\int_x^1 dz\,\frac{G\left(\frac{x}{z}\right)-G(x)}{1-z}+\left(\frac{11}{12}-\frac{1}{3}\frac{T(R)}{C_2(G)}+\ln(1-x)\right)G(x,t)\right]\right\}}$$