Lecture5

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Yang-Mills Theory

A. SU(N) Groups

QED Lagrangian has local Gauge Invariance for

$$\boxed{\begin{aligned} \psi' &\to e^{i\omega(x)}\psi(x) \\ A' &= A_\mu - \frac{1}{q}\partial_\mu \omega(x) \end{aligned}}$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(x)\,[\gamma_\mu(i\partial_\mu - qA_\mu - m)\psi]$$

$$= -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}(x)\,[i\gamma_\mu D_\mu - m]\psi$$

$$\boxed{D_\mu = \partial_\mu + iqA_\mu}$$

$$D'_\mu \psi' = \left[\partial_\mu + qi\left[A_\mu - \frac{1}{q}\partial_\mu \omega(x)\right]\right] e^{i\omega(x)}\psi(x)$$

$$e^{i\omega(x)}\psi(x)\cancel{[i\partial_\mu \omega]} + e^{i\omega(x)}\partial_\mu \psi(x)$$

$$+ iqA_\mu e^{i\omega x} - i\cancel{\partial_\mu \omega}\, e^{i\omega(x)}\psi(x) =$$

$$= e^{i\omega(x)} D_\mu(x)$$

$$\mathcal{L}' = \mathcal{L}$$

- $U(1)$ Transformation

$\psi' = e^{i w(x)} \psi(x)$ - is an Abelian transformation

$$\psi'' = e^{i\beta(x)} \psi'(x) = e^{i\beta(x)} e^{i w(x)} \psi(x)$$

$$= e^{i w(x)} e^{i\beta(x)} \psi(x)$$

$$\boxed{U(w(x))\, U(\beta(x)) = U(\beta(x))\, U(w(x))}$$

- Non Abelian Local Gauge Invariant field:

We consider now fields which are invariant with respect to Non Abelian Transformation

- These field were introduced by Yang-Mills - 1954

Yang-Mills fields

$\Rightarrow$ Instead of $\psi \to e^{iw}\psi$

Consider $\psi \to S\psi$

$SU(N)$ Transformation

where S - is a SU(N) transformation

- In this case ψ - should be as minimum N-row column $\psi = \begin{bmatrix} \psi^1 \\ \psi^2 \\ \vdots \\ \psi^N \end{bmatrix}$

Fundamental - Representation

- Transformation is described as

$$\psi \to S\psi$$

unitary $S^+S = SS^+$

unimodular $\det S = 1$

$N \times N$ - matrix

- unitarity means that $\psi'^+\psi' = \psi^+\psi = \sum_{1}^{N} \psi^{j*}\psi^j$

- unimodularity fixes the total phase

from $S^+S = SS^+ \Rightarrow |\det S| = 1$.

One may still have $S' \to Se^{i\omega}$.

but $\det S = 1$ fixes $\omega = 0$

Therefore no U(1) symmetry is allowed

- The $N \times N$ matrix with the conditions

$S^+S = SS^+ = 1 \qquad \det S = 1$

is characterized by N^2-1 independent parameters w^α $\alpha=1,2\ldots N^2-1$

$$\left(N\times N = 2N^2, \quad \begin{matrix} S^\dagger S = SS^\dagger = 1 \\ \frac{2N^3}{2} = N^2 \\ \det S = 1 \to N^2-1 \end{matrix} \right)$$

- it is always possible to choose w^α such that any S can be represented as

$$\boxed{S = e^{iw^\alpha t^a}}$$

where t^a – are fixed collection of N^2-1 independent matrices

- from the condition of $S^\dagger S = SS^\dagger = 1$ one obtains $(t^a)^\dagger = t^a$ Hermitean

- from the condition $\det S = 0$

$$\mathrm{Tr}\, t^a = 0$$

- t^a – represents a complete set

$$\mathrm{Tr}\, t^a t^b = \mathrm{Const}\, \delta^{ab}$$

- one can always renormalize t^a such that $\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$

- ... matrix $F = \sum C_a T_a$

— For any matrix

$$C_a = 2\,\mathrm{Tr}\, t^a F$$

— Therefore $\boxed{F_K{}^i = 2\,(t^a)^p{}_q\, F^q{}_p\, (t_a)^i{}_K}$

— Consider a particular case

$$(F^e{}_m)^i{}_k = \delta^i{}_m \delta^e{}_K - \frac{1}{N}\delta_k{}^i \delta^e{}_m$$

($e\,m$ is the identifier of the matrix)

$$\delta^i{}_m \delta^e{}_k - \frac{1}{N}\delta_k{}^i \delta^e{}_m =$$

$$2\,(t^a)^p{}_q \left(\delta^q{}_m \delta^e{}_p - \frac{1}{N}\delta^q{}_p \delta^e{}_m \right)(t_a)^i{}_k =$$

$$= 2\left[(t^a)^e{}_m - \frac{1}{N}\underbrace{(t^a)^p{}_p}_{0}\delta^e{}_m \right](t_a)^i{}_k =$$

$$= 2\,(t^a)^e{}_m\,(t^a)^i{}_k$$

$$\boxed{(t^a)^e{}_m\,(t^a)^i{}_k = \frac{1}{2}\delta^i{}_m \delta^e{}_k - \frac{1}{2N}\delta^i{}_k \delta^e{}_m}$$

Fierz - Identity (*)

— from above

$$(t^a)^e{}_m\,(t^a)^m{}_k = \frac{1}{2}N\delta^e{}_k - \frac{1}{2N}\delta^e{}_k$$

$$\boxed{t^a \cdot t^a = \frac{N^2-1}{2N} I}$$

— using $\mathrm{Tr}\, t^a t^b = \frac{1}{2}\delta^{ab}$ and F. Ident. (+)

$$tr(t^a t^b t^a t^c) =$$

$$tr\left((t^a)^e_m (t^b)^m_i (t^a)^i_k (t_c)^k_e\right) =$$

$$tr\left[(t^b)^m_i (t_c)^k_e \left[\frac{1}{2}\delta^i_m \delta^e_k - \frac{1}{2N}\delta^i_k \delta^e_m\right]\right]$$

$$= \frac{T}{2} r\left[\frac{1}{2}\underbrace{(t^b)^m_m}_{=0} (t_c)^k_k\right] - \frac{Tr}{2N}\left\{(t^b)^m_k (t_c)^k_m\right\}$$

$$= -\frac{1}{2N} Tr(t^b t_c) = -\frac{1}{4N}\delta^{bc}$$

$$\boxed{tr(t^a t^b t^a t^c) = -\frac{1}{4N}\delta^{bc}}$$ (++)

⇒ Commutator of t matrices

$$[t^a, t^b] = t^a t^b - t^b t^a$$

$$[t^a t^b]^\dagger = t^b t^a - t^a t^b = -[t^a t^b]$$

anti Hermitean

- also $Tr[t^a t^b] = 0$

- Thus one can express it through $\underbrace{i\, t^c}_{\text{anti Hermitean}}$

$$\boxed{[t^a t^b] = i f^{abc} t^c} = F = C_c t^c$$

(*)

- from condition $\quad C_c = 2Tr(t^c F)$

$$i f^{abc} = 2Tr([t^a t^b] t^c)$$

$$\boxed{f^{abc} = -i2Tr([t^a t^b] t^c)}$$

Structure constants

real and antisymmetric

- Using Notation (*)

$$Tr(t^a t^b t^a t^c) = Tr(t^a t^a t^b t^c) +$$

$$+ Tr[t^a [t^b, t^a] t^c] =$$

$$= t^a t^a t^b t^c \quad i f^{bae} T t^a t^e t^c =$$

$$= \mathrm{Tr}\, t\, t\, t\, t - i f \quad \mathrm{Tr}\, t\, t\, t$$

$$= \mathrm{Tr}\, t^a t^a t^b t^c - i \frac{f^{abe}}{2} \mathrm{Tr}\, t^a t^e t^c$$

$$- i \frac{f^{abe}}{2} \mathrm{Tr}\, t^a t^e t^c =$$

$$= \mathrm{Tr}\, t^a t^a t^b t^c - i \frac{f^{abe}}{2} \mathrm{Tr}\, t^a t^e t^c + \frac{i f^{eba}}{2} \mathrm{Tr}\, t^a t^e t^c$$

$e \to a$

$a \to e$

$\frac{i}{2} f^{abe} \mathrm{Tr}\, t^e t^a t^c$

$$= \mathrm{Tr}\, t^a t^a t^b t^c - i \frac{f^{abe}}{2} \mathrm{Tr}\, [t^a t^e] t^c =$$

$$= \mathrm{Tr}\, t^a t^a t^b t^c + \frac{f^{abe} f^{aed}}{2} \mathrm{Tr}\, t^d t^c$$

$$= \frac{N^2-1}{2N} \mathrm{Tr}\, t^b t^c + \frac{f^{abe} f^{aed}}{2} \mathrm{tr}\, t^d t^c$$

aec

$$= \frac{N^2-1}{4N} \delta^{bc} + \frac{f^{abe} f^{aed}}{4} \delta^{dc}$$

$$= \frac{N^2-1}{4N} \delta^{bc} - \frac{f^{abe} f^{ace}}{4}$$

$$\boxed{\mathrm{Tr}\, t^a t^b t^a t^c = \frac{(N^2-1)}{4N} \delta^{bc} - \frac{1}{4} f^{abe} f^{ace}}$$

$4N$

- On the other side from (††) one has

$$Tr\, t^a t^b t^a t^c = -\frac{1}{4N}\delta^{bc}$$

- Therefore $-\frac{1}{4N}\delta^{bc} = \frac{(N^2-1)}{4N}\delta^{bc} - \frac{1}{4}f^{abe}f^{ace}$

$$\Rightarrow \frac{1}{4}f^{abc}f^{ace} = \frac{N^2}{4N}\delta^{bc}$$

$$f^{abe}f^{ace} = N\delta^{bc}$$

$$f^{aeb}f^{aec} = N\delta^{bc}$$

$$\boxed{f^{abc}f^{abd} = N\delta^{cd}}$$

⇒ Some Notations

$$C_2(F) = C_F = \frac{N^2-1}{2N}$$

$$C_2(G) = C_V = N$$

⇒ Other Relations

$\Rightarrow$ Other Relations

$$\boxed{(t^a)^i_j \, (t^a)^j_k = C_F \, \delta^i_k = \frac{N^2-1}{2N} \delta^i_k}$$

using $(t^a)^e_m \, (t^a)^i_k = \frac{1}{2} \delta^i_m \, \delta^e_k - \frac{1}{2N} \delta^i_k \, \delta^e_m$

$$(t^a)^e_m \, (t^a)^m_k = \frac{N}{2} \delta^e_k - \frac{1}{2N} \delta^e_k = \frac{N^2-1}{N} I$$

$$\boxed{f^{abc} \, f^{abd} = C_V \, \delta^{cd} = N \, \delta^{cd}}$$

$$\Rightarrow \boxed{(t^a)^i_k \, (t^a)^e_m = \frac{1}{2} \delta^i_m \, \delta^e_k - \frac{1}{2N} \delta^i_k \, \delta^e_m}$$

$$\Rightarrow t^a t^b t^a = (t^a)^i_k \, (t^b)^k_e \, (t^a)^e_m =$$

$$= (t^b)^k_e \left[\frac{1}{2} \delta^i_m \, \delta^k_e - \frac{1}{2N} \delta^i_k \, \delta^e_m \right]$$

$$= (t^b)^k_k \frac{1}{2} \delta^i_m - \frac{1}{2N} (t^b)^i_m = -\frac{1}{2N} (t^b)^i_m$$

$$\boxed{t^a t^b t^a = -\frac{1}{2N} t^b}$$

$$\Rightarrow \quad i f^{abc} t^b t^c = \frac{i}{2} f^{abc} t^b t^c + \frac{i}{2} f^{abc} t^b t^c$$

$$= \frac{i}{2}\left(f^{abc} t^b t^c - f^{abc} t^c t^b \right) =$$

$$= \frac{i}{2} f^{abc} [t^b t^c] = -\frac{1}{2} f^{abc} f^{bce} t^e$$

$$= -\frac{1}{2} \underbrace{f^{bca} f^{bce}}_{N \delta^{ae}} t^e = -\frac{N}{2} t^e$$

$$\boxed{i f^{abc} t^b t^c = -\frac{N}{2} t^a}$$

$$\Rightarrow \quad f^{adg} f^{bed} f^{cge} = -\frac{N}{2} f^{abc} =$$

$$= -f^{agd} f^{bed} f^{cge} =$$

$$= 2i \operatorname{Tr}\left[[t^a t^g] t^d \right] \left(-2i \operatorname{Tr}\left[[t^b t^e], t^d \right] \right) f^{cge}$$

$$4 \; [t^a t^g]^m \; (t^d)^l \quad [t^b t^e]^k \; (t^d)^i_l \quad f^{cge}$$

$$= 4 \left[[t^a t^g]^m{}_e \, [t^b t^e]^k{}_i \left(\frac{1}{2} \delta^i_m \delta^e_k - \frac{1}{2N} \delta^i_k \delta^e_m \right) \right. f^{cge}$$

$$= 4 \left[[t^a t^g]^i{}_e \, [t^b t^e]^e{}_i - \frac{1}{2N} [t^a t^g]^m{}_m \, [t^b t^e]^k{}_k \right] f^{cge}$$

$$= 4 \, Tr [t^a t^g] [t^b t^e] \cdot f^{cge}$$

$$= -4 \, Tr [f^{agc} t^c f^{bek} t^k] \cdot f^{cge}$$

$$= -4 f^{agc} f^{bek} Tr \, t^c t^k \cdot f^{cge} =$$

$$=$$

$$f^{adg} f^{bed} f^{cge} = -\frac{N}{2} f^{abe}$$

$$f^{abc} f^{ade} f^{bdf} f^{ceg} = \frac{N^3}{2} \delta^{fg}$$