ALGEBRA AND TRIGONOMETRY REVIEW

by Dr TEBOU, FIU

A. Fundamental identities

Throughout this section, a and b denotes arbitrary real numbers.

i) **Square of a sum:** $(a+b)^2=a^2+2ab+b^2$
ii) **Square of a difference:** $(a-b)^2=$
iii) **Difference of squares:** $a^2-b^2=(a-b)(a+b)$
iv) **Sum of cubes:** $a^3+b^3=(a+b)(a^2-ab+b^2)$
v) **Difference of cubes:** $a^3-b^3=$

Exercises. 1) Use i) to complete ii) and use iv) to complete v); explain your reasoning.

2) Use i)-v) to factor each expression. a) x^4-y^4, b) $8x^3-27$, c) $4x^4-1$, d) $4z^2-12z+9$,

e) $9x^2-8$.

B. Factoring.

B1. Quadratic expression. Consider the quadratic expression: ax^2+bx+c, where a, b, c are constants that do not depend on x, and $a\neq 0$. We may then rewrite this expression as

$a(x^2+\frac{b}{a}x+\frac{c}{a})=a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right)$, by noting that $x^2+\frac{b}{a}x=x^2+2\frac{b}{2a}x$, and using i), one can easily complete the beginning of the square to get a complete square. Now, when we complete the square, we add an extra term, and so we should also subtract it to keep the given expression unchanged. So the original expression may be written, (thanks to i)):

$ax^2+bx+c= a\left(\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2-4ca}{4a^2}\right)\right)$. Set $\Delta=b^2-4ac$, which is called discriminant of the quadratic expression. Note that if $\Delta<0$, then the expression in the parentheses is always positive and the quadratic polynomial cannot be factored. If on the contrary, $\Delta>0$, then we can use the difference of squares formula to show that the quadratic polynomial has two distinct roots given by $x^+=\frac{-b+\sqrt{\Delta}}{2a}$ and $x^-=\frac{-b-\sqrt{\Delta}}{2a}$; thus $ax^2+bx+c=a(x-x^+)(x-x^-)$. Finally, for $\Delta=0$, the quadratic polynomial has a single double root $x= -\frac{b}{2a}$.

B2. Cubic and higher order expressions. We have just seen that there is a simple rule that may be used to determine whether a quadratic polynomial can be factored or not, and when it can be factored, the rule, known as the quadratic formula, also helps us get the roots of the polynomial. Now for cubic and higher order polynomials, we do not have such a simple rule. To factor such a polynomial, we will have to use the trial and error method to get one of its roots, then use division to reduce the degree of the polynomial to be factored. To use the trial and error method, evaluate the given polynomial at the number x=x_0 with x_0= 0, 1, -1, 2, -2, 3, -3, … until you get the value zero. Then x-x_0 is a factor of the given polynomial. Now divide the polynomial by x-x_0, you then get a polynomial of a lesser degree. If the degree is 3 or more, use again the trial and error method until you get a quadratic polynomial.

Note however that sometimes, just using the grouping method can save you all the work needed to carry out the trial and error method. So, think out of the box!

Exercises. Factor each expression: a) $f(x)=6x^4+x^3-24x^2-9x+10$, b) $g(x)=x^5-4x^3-27x^2+108$

c) $h(x)=4x^3-4x^2-3x+3$, d) $k(x)= 6x^3+17x^2+6x-8$, e) $m(x)=18x^4-9x^3-31x^2+2x+6$.

C. **Rationalizing.** When you want to <u>rationalize the denominator</u> of an expression involving a radical, multiply both the numerator and denominator by the conjugate of the denominator, and <u>expand the denominator only</u>; if the denominator is, say, $x-\sqrt{y}$, then multiply both numerator and denominator by $x+\sqrt{y}$, as $x+\sqrt{y}$ is the conjugate of $x-\sqrt{y}$. To <u>rationalize the numerator</u>, you proceed exactly the same way, and <u>expand the numerator only</u>.

Exercises. 1) Rationalize the numerator and simplify each expression: a) $r(x) = \frac{3-\sqrt{x+7}}{x^2-5x+6}$

b) $q(x) = \frac{\sqrt{2x+22}-4}{x^2+7x+12}$

2) Rationalize the denominator and simplify each expression c) $p(x) = \frac{x^2-x-2}{1-\sqrt{5x+6}}$

b) $n(x) = \frac{x^2-x-12}{6-\sqrt{5x+16}}$

D. **Functions.** A f*unction* is a rule that assigns to each admissible input exactly one output. A simple example of a function is the assignment to each FIU student or staff a panthersoft identity number (PID). The *domain* of a function is the set of all admissible inputs; for instance in the case of PID assignment, the domain of that function consists of all the staff and students of FIU; there are people in Miami who have no PID. The range of a function is the set of all the corresponding outputs. In the case of PID assignments, the range is the set of all assigned PIDs. The type of functions that we will be dealing with will be functions defined from the set of real numbers into itself; examples of functions include: polynomial functions whose domain is the whole set of real numbers, rational functions, a rational function is the ratio of two polynomial functions, if r(x)=$\frac{p(x)}{q(x)}$, where p and q are polynomial functions, then the domain of r, denoted D_r, consists of all real numbers x with q(x)≠0. Another type of function is the *absolute value function* defined by

$$|x| = \begin{cases} -x, \text{if } x < 0 \\ x, \text{if } x \geq 0. \end{cases}$$

A *piecewise defined function* is a function given by different formulas in different intervals; the absolute value function is an example of such a function.

Example: write the following function $h(x) = |-2x + 5| - |3x + 7|$ without the absolute value symbol. This amounts to writing the function *h* in piecewise defined form; to do this, which intervals are we going to use? To find the intervals, solve for *x* each of the arguments in the absolute value symbol: we shall solve for *x*:

-2x+5=0, getting x=5/2, and 3x+7=0, getting x=-7/3. Therefore our intervals are: $\left(-\infty, -\frac{7}{3}\right)$, $\left[-\frac{7}{3}, \frac{5}{2}\right)$, $\left[\frac{5}{2}, \infty\right)$. Thus, we have

$$h(x) = \begin{cases} -2x + 5 - \left(-(3x + 7)\right), \text{ if } x < -7/3 \\ -2x + 5 - (3x + 7), \text{ if } -\frac{7}{3} \leq x < 5/2 \\ -(-2x + 5) - (3x + 7), \text{ if } x \geq 5/2. \end{cases}$$

Some simple algebra yields

$$h(x) = \begin{cases} x + 12, \text{ if } x < -7/3 \\ -5x - 2, \text{ if } -\frac{7}{3} \leq x < 5/2 \\ -x - 12, \text{ if } x \geq \frac{5}{2}. \end{cases}$$

Exercises. 1) Find the domain of each function: a) f(x)=$\frac{x^2+5x+6}{x^2-x-12}$ b) g(x)=$\sqrt{x^3 - 2x^2 + 1}$

2) Write each function in a piecewise defined form: c) $k(x) = |6 - 5x| - |7x + 4|$, d) $m(x) = |4x - 9| - |11 - 6x|$.

Note. The sum, difference, product and quotient of two functions is a function.

Composition. The composition of a function f with a function g is the function fog given by $fog(x)=f(g(x))$ with domain $D_{fog}=\{x \text{ in } D_g;\ g(x) \text{ in } D_f\}$.

Example. If $f(x)=\sqrt{1-x}$ and $g(x)=\frac{2x-3}{4x+5}$, find fog and gof as well as D_{fog} and D_{gof}.

$fog(x)=\sqrt{1-g(x)}=\sqrt{1-\frac{2x-3}{4x+5}}=\sqrt{\frac{2x+8}{4x+5}}$; for $fog(x)$ to be defined, $g(x)$ must be defined, so

$x \neq -5/4$, and $\frac{2x+8}{4x+5} \geq 0$. To find where the ratio is nonnegative, we shall use a sign line.

$-\infty$ + -4 - -5/4 + ∞

The sign line shows that the ratio is nonnegative in the intervals $(-\infty,-4]$ and $(-5/4, \infty)$; therefore

$D_{fog(x)}=(-\infty,-4]\cup(-5/4, \infty)$. To get the correct signs, pick any number in each interval and plug it in the ratio, evaluate; the sign of the number that you obtain is the sign of the ratio in the corresponding interval. The case of gof is left as an exercise.

Inverse functions. Let f and g be two functions such that

$$g(f(x))=x, \text{ for all } x \text{ in the domain of } f, \text{ and}$$

$$f(g(y))=y, \text{ for every } y \text{ in the domain of } g,$$

Then f and g are called inverse functions; f is called the inverse of g, and g is called the inverse of f. The inverse function of a function f is denoted f^{-1}, and we have $D_f=R_{f^{-1}}$ and $R_f=D_{f^{-1}}$.

Example. Let $(x)=\frac{-3x+8}{5x+4}$. Find $f^{-1}(x)$, and D_f . To find $f^{-1}(x)$, set $y=\frac{-3x+8}{5x+4}$, switch x and y to get $x=\frac{-3y+8}{5y+4}$. Solve the equation for y in terms of x to get $f^{-1}(x)$. It follows from the equation

$x(5y+4)=-3y+8$. Passing y from the right hand side to the left, and factoring, we get

$y(5x+3)=8-4x$. Consequently, it follows from the last equation, $y=\frac{8-4x}{5x+3}$. Hence,

$f^{-1}(x)=\frac{8-4x}{5x+3}$ and $R_f=\{x; x \neq -3/5\}=(-\infty,-\frac{3}{5})\cup(-\frac{3}{5},\infty)$. As an exercise, do the same for $f(x)=\sqrt{3x-4}$.

Note. A function f is called one to one if $f(x) \neq f(z)$ whenever $x \neq z$. Any one to one function f defined from its domain to its range has an inverse, or is called invertible. The graph of an invertible function and that of its inverse are symmetric about the line $y=x$.

Difference quotient. Let f be a function. We define the *difference quotient* of f at x to be the ratio

$\frac{f(x+h)-f(x)}{h}$ where h is a nonzero constant independent of x.

Example. If $f(x) = 2x^3 + x - 1$, find the difference quotient of *f*. We have, by using the difference of cubes formula:

$\frac{f(x+h)-f(x)}{h} = \frac{2(x+h)^3+(\cancel{x}+h)-\cancel{1}-(2x^3+\cancel{x}-\cancel{1})}{h} = \frac{2((x+h)^3-x^3)+h}{h} = \frac{2((\cancel{x}+h-\cancel{x})((x+h)^2+(x+h)x+x^2)+h}{h} = 2((x+h)^2 + (x+h)x + x^2) + 1$, in the last step, h was factored out in the numerator, and cancelled with *h* in the denominator. Remember that in order to get $f(x+h)$, you replace *x* with *x+h* everywhere in the expression of $f(x)$.

Exercises. If $f(x) = 3x^2 - 7x + 4$ and $g(x) = \frac{2x+8}{4x+5}$, find the difference quotients of *f* and *g* at *x*. Be sure to simplify your answers as much as possible.

Parity. A function *f* is called *even* if *f(-x)=f(x)* for every *x* in the domain of *f* . For instance, if $f(x)=x^2$ or *f(x)*= cos(*x*), then *f* is even, as $(-x)^2=x^2$ and cos(-*x*)=cos(*x*), for every real number *x.*

A function *f* is called *odd* if *f(-x)=-f(x)* for each *x* in the domain of *f* . If $f(x)=x^3$ or *f(x)*= sin(*x*), then *f* is odd, as $(-x)^3= - x^3$ and sin(-*x*)= - sin(*x*), for every real number *x.*

E. Exponential and Logarithmic Functions.

E1. Exponential Functions. Let *a>0* be a real number. The *exponential function* with base *a* is the function *f* defined by $f(x)=a^x$ for every real number *x*. The domain and range of *f* are given by $D_f = (-\infty, \infty)$, and $R_f = (0, \infty)$. So exponential functions are defined everywhere, and are never zero and never negative.

Properties. Let *a>0*, *x*, and *y*, be real numbers. Then

i) $a^{x+y}=a^x a^y$
ii) $a^{x-y} = a^x/a^y$
iii) $a^{xy}=(a^x)^y=(a^y)^x$
iv) $a^0=1$

Note. Among all the exponential functions there is one called *natural exponential function* whose base denoted *e* is a number between 2 and 3; more precisely, $e \cong 2.71828$1828459045235. Any time that the base of an exponential function is not specified, you should understand that the base is *e*.

E2. Logarithmic functions. Let *b>0* and $b \neq 1$. The *logarithmic function* with base *b* is the function denoted by $\log_b$ and defined by $y = \log_b x$ if and only if $x = b^y$.

Note. i) It follows from this definition that the exponential function with base *b* and the logarithmic function with base *b* are inverse functions. Therefore, the domain of $\log_b$ is $(0, \infty)$, and the range of $\log_b$ is $(-\infty, \infty)$. Further,

for every real number *x*, we have $\log_b(b^x) = x$, and for every *x*>*0*, we have $b^{\log_b(x)} = x$.

ii) The logarithmic function with base *e* is called natural logarithm and denoted ln while the logarithmic function with base 10 is called common logarithm and denoted log.

iii) (Change of base formula) Let *b*>*0* and $b \neq 1$ and let *a*>*0* and $a \neq 1$. Let *x*>*0*. Then $\log_b(x) = \frac{\log_a(x)}{\log_a(b)} = \frac{\ln(x)}{\ln(b)}$. As an exercise, prove the change of base formula.

Properties. Let *b*>*0* and $b \neq 1$, and let *x*>*0*, *y*>*0*, and r be real numbers.

i) $\log_b(xy) = \log_b(x) + \log_b(y)$

ii) $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

iii) $\log_b(x^r) = r\log_b(x)$

iv) $\log_b(1) = 0$

v) $\log_b(b) = 1$.

Exercises. 1) Find the domain of each function: a) $f(x) = \log_x(-3x + 5)$,

b) $g(x) = \log_3(x^2 - x - 1)$. 2) Use the properties of exponential functions prove each of the properties i) to v).

3) Expand the logarithm in terms of sums, differences and multiples of simpler logarithms

a) $f(x) = \log_2 \frac{\sqrt[5]{7x-9}}{\sin 3x \sec 2x}$ b) $h(x) = \log_5 \frac{x^3\sqrt{x^2+2x-2}}{(3x-8)(2\cot x+5)}$

4) Solve each equation for *x*. a) $\log_x\left(-3x\sqrt{2} + 8\right) = 2$, b) $\ln(4x) - 3\ln(x^2) = \ln 2$,

c) $e^{-2x} - 3e^{-x} + 2 = 0$, d) $x^{(2\ln x - 3)} = 4$.

F. Algebra of infinity. It is important to always keep in mind that $-\infty$ and $+\infty$ are symbols and not numbers. Accordingly, the algebra involving these symbols is somehow different from the algebra of real numbers. We have

i) $+\infty + \infty = +\infty$ and $-\infty + (-\infty) = -\infty$

ii) $+\infty(+\infty) = +\infty$ and $-\infty(+\infty) = -\infty$, and $-\infty(-\infty) = +\infty$

iii) For every real number *a*, we have $a + (-\infty) = -\infty$, $a - (-\infty) = +\infty$, and $a/\pm\infty = 0$.

iv) For every nonzero real number *a*, we have $a(-\infty) = \begin{cases} +\infty \text{ if } a < 0 \\ -\infty \ if\ a > 0. \end{cases}$

v) **Indeterminate forms:** These are quantities which do not have a definite value; they are: a) $+\infty - (+\infty)$ or $\infty - \infty$, *b*) $\frac{\pm\infty}{\pm\infty}$, c) $0(\pm\infty)$, d) $1^{\pm\infty}$, e) ∞^0, f) 0/0, g) 0^0. Any time that you encounter one of those indeterminate forms, you should work out to get rid of them; many examples will be discussed in class.

G. Trigonometric Functions. The six fundamental trigonometric functions are stated in the chart with their domain and range. All angles are in radian. You have the abbreviation of each function into parentheses. If you do not remember, check their graphs in the textbook.

Function	**Domain**	**Range**
Cosine (cos)	$(-\infty,+\infty)$	$[-1,1]$
Sine (sin)	$(-\infty,+\infty)$	$[-1,1]$
Tangent (tan) tan x=$\frac{\sin x}{\cos x}$	$\{x; x \neq \frac{\boldsymbol{\pi}}{\mathbf{2}}$ +kπ, k is an integer$\}$	$(-\infty,+\infty)$
Cotangent (cot) cot x=$\frac{\cos x}{\sin x} = \frac{1}{tanx}$	$\{x; x \neq \boldsymbol{\pi}$ +kπ, k is an integer$\}$	$(-\infty,+\infty)$
Secant (sec) sec x=$\frac{1}{\cos x}$	$\{x; x \neq \frac{\boldsymbol{\pi}}{\mathbf{2}}$ +kπ, k is an integer$\}$	$(-\infty,-1] \cup [1,+\infty)$
Cosecant (csc) csc x=$\frac{1}{\sin x}$	$\{x; x \neq \boldsymbol{\pi}$ +kπ, k is an integer$\}$	$(-\infty,-1] \cup [1,+\infty)$

G1. Important Trigonometric Identities.

Pythagorean identities:

a) $\cos^2 x+\sin^2 x=1$, for every real number x
b) $1+\tan^2 x=\sec^2 x$, for every x in D_{tan}
c) $1+\cot^2 x=\csc^2 x$, for every x in D_{cot}

Complement identities

$\cos x = \sin(\frac{\pi}{2} - x)$ and $\sin x = \cos(\frac{\pi}{2} - x)$, for every number x.

Supplement identities

$\sin x = \sin(\pi - x)$, and $\cos(\pi - x) = -\cos x$, for every number x.

Sum formulas

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\sin(x + y) = \sin x \cos y + \cos x \sin y$, for all numbers x and y.

Exercise. Use the sum formulas to find:

$\cos(x - y) =$ $\tan(x + y) =$

$\sin(x - y) =$ $\tan(x - y) =$

Double-angle formulas

$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2(x) - 1 = 1 - 2\sin^2 x$, for every number x.

$\sin(2x) = 2 \sin x \cos x$, for every number x.

Exercise. Show that $\cos^4 x - \sin^4 x = \cos^2 x - \sin^2 x$, for every number x.

Product to sum formulas

$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$

$2 \cos x \sin y = \sin(x + y) + \sin(y - x)$

$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$

Trigonometric values to know

x	$\cos x$	$\sin x$	$\tan x$	$\sec x$	$\csc x$	$\cot x$
0	1	0	0	1	undefined	undefined
$\pi/6$	$\sqrt{3}/2$	1/2	$1/\sqrt{3}$	$2/\sqrt{3}$	2	$\sqrt{3}$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	$\sqrt{2}$	$\sqrt{2}$	1
$\pi/3$	1/2	$\sqrt{3}/2$	$\sqrt{3}$	2	$2/\sqrt{3}$	$1/\sqrt{3}$
$\pi/2$	0	1	undefined	undefined	1	0
π	-1	0	0	-1	undefined	undefined

G2. Inverse trigonometric functions. Given that none of the six trigonometric functions in the chart above is one-to-one on its domain, in order to be able to define an inverse for any of those functions, we shall restrict its domain to a subdomain where it is one-to-one.

Inverse cosine function. The inverse cosine function denoted $\cos^{-1}$ is defined for every x in $[-1,1]$ by: $y = \cos^{-1} x$ if and only if $x = \cos y$, y in $[0, \pi]$.

Inverse sine function. The inverse sine function denoted $\sin^{-1}$ is defined for every x in $[-1,1]$ by: $y = \sin^{-1} x$ if and only if $x = \sin y$, y in $[-\pi/2, \pi/2]$.

Inverse tangent function. The inverse tangent function denoted $\tan^{-1}$ is defined for every x in $(-\infty,\infty)$ by: $y=\tan^{-1}x$ if and only if $x=\tan y$, y in $(-\frac{\pi}{2},\frac{\pi}{2})$.

Inverse secant function. The inverse cosine function denoted $\sec^{-1}$ is defined for every x in $(-\infty,-1]\cup[1,\infty)$ by: $y=\sec^{-1}x$ if and only if $x=\sec y$, y in $[0,\frac{\pi}{2})\cup(\pi/2,\pi]$.

Inverse cosecant function. The inverse cosecant function denoted $\csc^{-1}$ is defined for every x in $(-\infty,-1]\cup[1,\infty)$ by: $y=\csc^{-1}x$ if and only if $x=\csc y$, y in $[-\frac{\pi}{2},0)\cup(0,\pi/2]$.

Inverse cotangent function. The inverse cotangent function denoted $\cot^{-1}$ is defined for every x in $(-\infty,\infty)$ by: $y=\cot^{-1}x$ if and only if $x=\cot y$, y in $(0,\pi)$.

Exercise. A) For what values of x do we have a) $\cos(\cos^{-1}x)=x$, b) $\cos^{-1}(\cos x)=x$

c) $\sin^{-1}(\sin x)=x$, d) $\sin(\sin^{-1}x)=x$, e) $\tan(\tan^{-1}x)=x$, f) $\tan^{-1}(\tan x)=x$.

B) Show that 1) $\sec^{-1}x=\cos^{-1}(1/x)$, 2) $\csc^{-1}x=\sin^{-1}(1/x)$, 3) $\cot^{-1}x=\tan^{-1}(1/x)$.

Some identities involving inverse trigonometric functions

- **i)** $\cos^{-1}x+\sin^{-1}x=\pi/2$
- **ii)** $\cos(\sin^{-1}x)=\sqrt{1-x^2}=\sin(\cos^{-1}x)$
- **iii)** $\tan(\cos^{-1}x)=\frac{\sqrt{1-x^2}}{x}$
- **iv)** $\tan(\sin^{-1}x)=\frac{x}{\sqrt{1-x^2}}$
- **v)** $\sec(\tan^{-1}x)=\sqrt{1+x^2}$
- **vi)** $\tan(\sec^{-1}x)=\frac{x}{|x|}\sqrt{x^2-1}$
- **vii)** $\tan^{-1}x+\cot^{-1}x=\frac{x}{|x|}\frac{\pi}{2}$

Exercise. Find a) $\cos(\tan^{-1}x)$, b) $\sin(\tan^{-1}x)$, c) $\cos(\sec^{-1}x)$, d) $\sin(\sec^{-1}x)$.