Assignment 1 - key

1.

a)

$$A_m = \begin{pmatrix} 1 & 2 & 3 & m \\ 2 & 1 & m & 3 \\ 3 & m & 1 & 2 \\ m & 3 & 2 & 1 \end{pmatrix}$$

b)

$$A_m \xrightarrow[\substack{-3r_1+r_3 \\ -mr_1+r_4}]{-2r_1+r_2} \begin{pmatrix} 1 & 2 & 3 & m \\ 0 & -3 & m-6 & 3-2m \\ 0 & m-6 & -8 & 2-3m \\ 0 & 3-2m & 2-3m & 1-m^2 \end{pmatrix} \xrightarrow[\left(\frac{3-2m}{3}\right)r_2+r_4]{\left(\frac{m-6}{3}\right)r_2+r_3}$$

$$\begin{pmatrix} 1 & 2 & 3 & m \\ 0 & -3 & m-6 & 3-2m \\ 0 & 0 & \frac{(m-6)^2}{3}-8 & \frac{(m-6)(3-2m)}{3}+2-3m \\ 0 & 0 & \frac{(m-6)(3-2m)}{3}+2-3m & \frac{(3-2m)^2}{3}+1-m^2 \end{pmatrix}$$

$$\det(A_m) = -3 \begin{vmatrix} \frac{(m-6)^2-24}{3} & \frac{(m-6)(3-2m)+3(2-3m)}{3} \\ \frac{(m-6)(3-2m)+3(2-3m)}{3} & \frac{(3-2m)^2+(1-m^2)(3)}{3} \end{vmatrix}$$

$$= -\frac{1}{3}\Big[\big((m-6)^2-24\big)\big((3-2m)^2+3(1-m^2)\big) - \big((m-6)(3-2m)+3(2-3m)\big)\big((m-6)(3-2m)+3(2-3m)\big)\Big]$$

$$= -\frac{1}{3}\Big[(m^2-12m+12)(m^2-12m+12) - (-2m^2+6m-12)^2\Big]$$

$$= -\frac{1}{3}\big(m^2-12m+12+(-2m^2+6m-12)\big)\big(m^2-12m+12-(-2m^2+6m-12)\big)$$

$$= -\frac{1}{3}(-m^2-6m)(3m^2-18m+24)$$

$$= m(m+6)(m^2-6m+8)$$

$$= m(m+6)(m-2)(m-4)$$

A_m is singular if $\det A_m = 0$, that is, for $m=0$, $m=-6$, $m=2$, or $m=4$

C. $m=0$.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -1 \\ 2 & 1 & 0 & 3 & 1 \\ 3 & 0 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 & 0 \end{array}\right) \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -1 \\ 0 & -3 & -6 & 3 & 3 \\ 0 & -6 & -8 & 2 & 3 \\ 0 & 3 & 2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[r_2+r_4]{-2r_2+r_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -1 \\ 0 & -3 & -6 & 3 & 3 \\ 0 & 0 & 4 & -4 & -3 \\ 0 & 0 & -4 & 4 & 3 \end{array}\right) \xrightarrow[-r_2/3]{r_3+r_4} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 0 & -1 \\ 0 & +1 & 2 & -1 & -1 \\ 0 & 0 & 4 & -4 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore

$$x_3 - x_4 = -\frac{3}{4} \quad \text{or} \quad x_3 = x_4 - 3/4$$

$$x_2 + 2x_3 - x_4 = -1 \to x_2 = -2\left(x_4 - \frac{3}{4}\right) + x_4 - 1 = -x_4 + \frac{1}{2}$$

$$x_1 + 2x_2 + 3x_3 = -1 \to x_1 = -2\left(-x_4 + \frac{1}{2}\right) - 3\left(x_4 - \frac{3}{4}\right) - 1$$

$$= -x_4 + \frac{1}{4}$$

$$S_0 = \left\{ \left(-\alpha + \frac{1}{4}, -\alpha + \frac{1}{2}, \alpha - \frac{3}{4}, \alpha\right)^T ; \alpha \in \mathbb{R} \right\}$$

$m=-6$.

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -6 & -7 \\ 2 & 1 & -6 & 3 & 1 \\ 3 & -6 & 1 & 2 & 0 \\ -6 & 3 & 2 & 1 & 0 \end{array}\right) \xrightarrow[\substack{-3r_1+r_3 \\ 6r_1+r_4}]{-2r_1+r_2}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -6 & -7 \\ 0 & -3 & -12 & 15 & 15 \\ 0 & -12 & -8 & 20 & 21 \\ 0 & 15 & 20 & -35 & -42 \end{array}\right) \xrightarrow[5r_2+r_4]{-4r_2+r_3}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -6 & -7 \\ 0 & -3 & -12 & 15 & 15 \\ 0 & 0 & 40 & -40 & -39 \\ 0 & 0 & -40 & 40 & 33 \end{array}\right) \xrightarrow{r_3+r_4}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -6 & -7 \\ 0 & -3 & -12 & 15 & 15 \\ 0 & 0 & 40 & -40 & -39 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right) \rightarrow 0 = -6$$ system is inconsistent

$S_6 = \emptyset \leftarrow$ empty set

$m = 2$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 2 & 1 & 2 & 3 & 1 \\ 3 & 2 & 1 & 2 & 0 \\ 2 & 3 & 2 & 1 & 0 \end{array}\right) \xrightarrow[\substack{-3r_1+r_3 \\ -2r_1+r_4}]{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 0 & -3 & -4 & -1 & -1 \\ 0 & -4 & -8 & -4 & -3 \\ 0 & -1 & -4 & -3 & -2 \end{array}\right)$$

$$\xrightarrow{r_2 \leftrightarrow r_4} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 0 & -1 & -4 & -3 & -2 \\ 0 & -4 & -8 & -4 & -3 \\ 0 & -3 & -4 & -1 & -1 \end{array}\right) \xrightarrow[-3r_2+r_4]{-4r_2+r_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 0 & -1 & -4 & -3 & -2 \\ 0 & 0 & 8 & 8 & 5 \\ 0 & 0 & 8 & 8 & 5 \end{array}\right)$$

$$\xrightarrow{-r_3+r_4} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 1 \\ 0 & -1 & -4 & -3 & -2 \\ 0 & 0 & 8 & 8 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$8x_3 + 8x_4 = 5 \rightarrow x_3 = -x_4 + \frac{5}{8}$

$-x_2 - 4x_3 - 3x_4 = -2$

$x_2 = 2 - 4\left(-x_4 + \frac{5}{8}\right) - 3x_4$

$\quad = x_4 - \frac{1}{2}$

$x_1 + 2x_2 + 3x_3 + 2x_4 = 1$

$x_1 = 1 - 2\left(x_4 - \frac{1}{2}\right) - 3\left(-x_4 + \frac{5}{8}\right) - 2x_4$

$\quad = -x_4 + \frac{1}{8}$

$S_2 = \left\{ \left(-\alpha + \frac{1}{8}, \alpha - \frac{1}{2}, -\alpha + \frac{5}{8}, \alpha\right)^T ; \alpha \in \mathbb{R} \right\}$

$m=4$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 2 & 1 & 4 & 3 & 1 \\ 3 & 4 & 1 & 2 & 0 \\ 4 & 3 & 2 & 1 & 0 \end{array}\right) \xrightarrow[\substack{-3r_1+r_3 \\ -4r_1+r_4}]{-2r_1+r_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & -3 & -2 & -5 & -5 \\ 0 & -2 & -8 & -10 & -9 \\ 0 & -5 & -10 & -15 & -12 \end{array}\right)$$

$$\xrightarrow{-\frac{r_3}{2} \leftrightarrow r_2} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 4 & 5 & 9/2 \\ 0 & -3 & -2 & -5 & -5 \\ 0 & -5 & -10 & -15 & -12 \end{array}\right) \xrightarrow[5r_2+r_4]{3r_2+r_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 4 & 5 & 9/2 \\ 0 & 0 & 10 & 10 & 17/2 \\ 0 & 0 & 10 & 10 & 21/2 \end{array}\right)$$

$$\xrightarrow{-r_3+r_4} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 3 \\ 0 & 1 & 4 & 5 & 9/2 \\ 0 & 0 & 10 & 10 & 17/2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right) \longrightarrow 0=2, \text{ system is inconsistent}$$

$S_4 = \phi$

2. $A \in M_n$, $B \in M_n$, A is nonsingular, $B^2 = O_{M_n}$. Show that $K = I_n + A^{-1}BA$ is nonsingular and find its inverse.

$$\begin{aligned} K^2 &= (I_n + A^{-1}BA)(I_n + A^{-1}BA) \\ &= I_n + A^{-1}BA + A^{-1}BA + (A^{-1}BA)(A^{-1}BA) \\ &= I_n + 2A^{-1}BA + A^{-1}B\underbrace{(AA^{-1})}_{I_n}BA \\ &= I_n + 2A^{-1}BA + A^{-1}(B^2)A \\ &= I_n + 2A^{-1}BA + \underbrace{A^{-1}(O_{M_n})A}_{O_{M_n}} \\ &= I_n + 2(K - I_n) \\ &= 2K - I_n \end{aligned}$$

So $2K - K^2 = I_n$ or $K(2I_n - K) = I_n$; hence K is invertible and $K^{-1} = 2I_n - K$.

3. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & -1 \\ -1 & 2 & 0 \end{pmatrix}$

Show that B is nonsingular. By Theorem 1.5.2 (ECN), it is enough to show that the RREF of B is I_3. We can compute B^{-1} as well by using the augmented matrix $(B \mid I_3)$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & -1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -2 & 1 & 0 \\ 0 & 4 & -1 & 1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\frac{4}{5}r_2+r_3} \left(\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -5 & 1 & -2 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{4}{5} & 1 \end{array}\right) \xrightarrow{\frac{2}{5}r_2+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{3}{5} & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & -5 & 1 & -2 & 1 & 0 \\ 0 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{4}{5} & 1 \end{array}\right)$$

$$\xrightarrow[5r_3+r_2]{3r_3+r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & -3 \\ 0 & -5 & 0 & -5 & 5 & 5 \\ 0 & 0 & -\frac{1}{5} & -\frac{3}{5} & \frac{4}{5} & 1 \end{array}\right) \xrightarrow[-5r_3]{-\frac{1}{5}r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -2 & -3 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 & -4 & -5 \end{array}\right)$$

Hence B is invertible and

$$B^{-1} = \begin{pmatrix} 2 & -2 & -3 \\ 1 & -1 & -1 \\ 3 & -4 & -5 \end{pmatrix}$$

4) $A \in M_{m,n}$ has a row consisting of zero entries only. $B \in M_{n,r}$.
$A \in M_{m,n}$ has a row consisting entirely of zeros.
Show that AB has a row consisting entirely of zeros. Set $AB = (d_{lk})$, $A = (a_{lk})$, $B = (b_{lk})$.
Say, the i^{th} row of A consists entirely of zeros.

Then $d_{lk} = \sum_{q=1}^{n} a_{lq} b_{qk}$; so $d_{ik} = \sum_{q=1}^{n} a_{iq} b_{qk}$.

Now $a_{iq} = 0$ for all $q = 1, 2, \ldots, n$, by hypothesis. Hence $d_{ik} = 0$ for all $k = 1, 2, \ldots, r$. Therefore AB has a row consisting entirely of zeros.