Simultaneous and indirect control of waves: some recent developments and open problems

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Overview

- An abstract model

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- Simultaneous stabilization

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  - Brief literature

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  - Lamé systems with localized damping

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- Indirect stabilization

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- Some extensions and open problems

Model formulation

Let H and V be Hilbert spaces with $V \subset H$. Assume V is dense in H and the injection of V into H is compact. Denote by $(.,.)$ the inner product in H, by $|.|$ the corresponding norm, and by V' the dual of V. Consider the damped abstract equation

$$\begin{array}{l} y_{tt} + Ay + By_t = 0 \text{ in } (0, \infty) \\ y(0) = y^0 \in V, \quad y_t(0) = y^1 \in H, \end{array}$$

where $A \in \mathcal{L}(V, V')$ is a selfadjoint coercive operator with $D(A^{\frac{1}{2}}) = V$, and $B \in \mathcal{L}(H)$ is a nonnegative operator.

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where $A \in \mathcal{L}(V, V')$ is a selfadjoint coercive operator with $D(A^{\frac{1}{2}}) = V$, and $B \in \mathcal{L}(H)$ is a nonnegative operator.

Introduce the energy

$$E(t) = \frac{1}{2}\{|y_t(t)|^2 + |A^{\frac{1}{2}}y(t)|^2\}, \quad \forall t \geq 0.$$

Theorem: Dafermos criterion

1970: Dafermos proves: the abstract system is strongly stable

$$\lim_{t\to\infty} E(t) = 0$$

if and only if

$$\mathrm{Ker}B \cap \mathrm{Ker}(A + \lambda I) = \{0\}, \quad \forall\lambda \in \mathbb{R}$$

where I denotes the identity operator on H.

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For the stabilization of single component systems, we refer to the contributions of Bardos-Lebeau-Rauch, Rauch-Taylor, Russell, Dafermos, Chen, Haraux, Komornik, Lasiecka, Nakao, Liu, Martinez, Triggiani, Zuazua,...

Brief literature

By simultaneous stabilization, we should understand stabilizing a multi-component system using the same damping mechanism in all components; the matrix defining the damping is degenerate.

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- 1986: Russell introduces the notion of simultaneous control for pdes when studying the boundary controllability of the Maxwell's equations.
- 1988: Lions (v.1, Controllability book) analyzes simultaneous boundary control problems for two uncoupled waves, and for two uncoupled plates.

Brief literature

Consider the system of uncoupled wave equations

$$
\begin{aligned}
&u_{jtt} - a_j \Delta u_j = 0 \text{ in } Q \\
&u_j = 0 \text{ on } \Gamma \times (0, T) \\
&u_j(x, 0) = u_j^0(x), \quad u_{jt}(x, 0) = u_j^1(x) \text{ in } \Omega, \quad j = 1,\ 2, ...,\ q,
\end{aligned}
$$

where $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$ for each j.

1988: Haraux (1988) shows for arbitrary nonempty open set ω:

- If $\sum_{j=1}^{q} u_j(x,t) = 0$ in $\omega \times (0,T)$ then $u_j^0 = 0, \quad u_j^1 = 0$ in $\Omega, \quad \forall j$.
provided that $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

- If $\sum_{j=1}^{q} u_j(x,t) = 0$ in $\omega \times (0,T)$ then $u_j^0 = 0, \quad u_j^1 = 0$ in $\Omega, \quad \forall j$.
  provided that $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

- If $N = 1$ and T is large enough, or $\omega = \Omega$, then there exists $C > 0$: for all j and all $(u_j^0, u_j^1) \in L^2(\Omega) \times H^{-1}(\Omega)$

$$\sum_{j=1}^{q} \{||u_j^0||^2_{L^2(\Omega)} + ||u_j^1||^2_{H^{-1}(\Omega)}\} \leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_j(x,t)|^2 \, dxdt$$

  provided that $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

(GCC): Bardos-Lebeau-Rauch (1992): ω is an admissible control region in time T if every ray of geometric optics enters ω in a time less than T.

Theorem 1 (CRAS, Paris, 2012)

Let T_0 denote the best controllability time for a single wave equation with unit speed of propagation. Suppose that $T > T_0 \max\{a_j^{-\frac{1}{2}}; j = 1,\ 2,\ \ldots,\ q\}$ and (ω, T) satisfies (GCC). There exists a constant $C > 0$ such that for all $(u_j^0, u_j^1) \in H_0^1(\Omega) \times L^2(\Omega)$, $j = 1,\ 2, \ldots,\ q$:

$$\sum_{j=1}^{q} \{||u_j^0||^2_{H_0^1(\Omega)} + ||u_j^1||^2_{L^2(\Omega)}\} \leq C \int_0^T \int_\omega |\sum_{j=1}^{q} u_{jt}(x,t)|^2\, dxdt,$$

with $C = C(\Omega, \omega, T, (a_j)_j, q)$, if and only if $a_j \neq a_k$ for all $j,\ k$ with $j \neq k$.

Lamé systems with localized damping

Given $(y_j^0, y_j^1)_j \in \left(\left[H_0^1(\Omega)\right]^N \times \left[L^2(\Omega)\right]^N\right)^q$, and a function $d \in L^\infty(\Omega)$, $d \geq 0$, consider the damped elastodynamic system

$$\begin{aligned}
&y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(y_j) + d \sum_{k=1}^{q} y_{kt} = 0 \text{ in } \Omega \times (0,\infty) \\
&y_j = 0 \text{ on } \Gamma \times (0,\infty) \\
&y_j(x,0) = y_j^0(x), \quad y_{jt}(x,0) = y_j^1(x), \text{ in } \Omega, \\
&j = 1,\ 2, ...,\ q,
\end{aligned}$$

where, for each j, μ_j and λ_j are the Lamé constants.
The total energy is given, for all $t \geq 0$, by

$$2E(t) = \sum_{j=1}^{q} \int_\Omega \{|y_{jt}(x,t)|^2 + \mu_j |\nabla y_j(x,t)|^2 + (\mu_j + \lambda_j)|\mathrm{div}(y_j(x,t))|^2\}\, dx$$

Lamé systems with localized damping

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_{\Omega} d(x) \left| \sum_{k=1}^{q} y_{kt}(x,t) \right|^2 dx.$$

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Question 1: Does the energy E decay to zero as time goes to infinity?

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Question 1: Does the energy E decay to zero as time goes to infinity?
Question 2: Under which conditions is the Lamé system exponentially stable?

Introduce the Hilbert space $\mathcal{H} = \left(\left[H_0^1(\Omega)\right]^N \times \left[L^2(\Omega)\right]^N\right)^q$ over the field $\mathbb{C}$ of complex numbers, equipped with the norm

$$||Z||_{\mathcal{H}}^2 = \sum_{j=1}^{q} \int_{\Omega} \{|v_j(x)|^2 + \mu_j |\nabla u_j(x)|^2 + (\mu_j + \lambda_j)|\text{div}(u(x))|^2\}\, dx,$$

$\forall Z = ((u_j, v_j)_j) \in \mathcal{H}$.

Set $Z_j = (y_j, y_{j,t})$. The Lamé system may be recast as the first order abstract evolution equation

$$\dot{Z}_j = \mathcal{A}_j Z_j, \quad Z_j(0) = (y_j^0, y_j^1), \;\; j = 1,\; 2,\; ...,\; q,$$

where the dot denotes differentiation with respect to time,and the unbounded operator $\mathcal{A}$ is given by

$$\mathcal{A}_j = \begin{pmatrix} 0 & I \\ \mu_j \Delta + (\mu_j + \lambda_j)\nabla \text{div} & -dL \end{pmatrix}$$

with $Lv = \sum_{j=1}^{q} v_j$, for every $v = (v_j)_j \in [L^2(\Omega)]^{Nq}$, and

$$\begin{aligned} D(\mathcal{A}_j) &= \Big\{(u_j, v_j) \in [H_0^1(\Omega)]^N \times [H_0^1(\Omega)]^N; \\ &\qquad \mu_j \Delta u_j + (\mu_j + \lambda_j)\nabla \text{div} u_j \in [L^2(\Omega)]^N\Big\}. \end{aligned}$$

It can be checked that one has (assuming for instance that Γ is C^2)

$$D(\mathcal{A}_j) = [H^2(\Omega) \cap H_0^1(\Omega)]^N \times [H_0^1(\Omega)]^N.$$

Thus, the operator $\mathcal{A}_j$ has a compact resolvent. Consequently the spectrum of $\mathcal{A}_j$ is discrete for each j.

With the help of Lumer-Phillips Theorem, (Pazy's book on semigroups, p. 14), one can show that the operator $\mathcal{A} = (\mathcal{A}_j)_j$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. Indeed, $D(\mathcal{A})$ is dense in $\mathcal{H}$, $\mathcal{A}$ is dissipative

$$\Re(\mathcal{A}Z, Z) = -\int_\Omega d(x)|\sum_{j=1}^{q} v_j(x)|^2\,dx \leq 0, \ \ \forall Z \in D(\mathcal{A}),$$

and (denoting by $\mathcal{I}$ the identity operator on $\mathcal{H}$):

$$R(\mathcal{I} - \mathcal{A}) = \mathcal{H}, \text{ by Lax-Milgram Lemma.}$$

Lamé systems with localized damping

Theorem 2: Strong stability (2018)

Let ω be a nonempty open subset of Ω. Suppose that d is positive in ω. The elastodynamic system is strongly stable:

$$\lim_{t\to\infty} E(t) = 0$$

if and only if the propagation speeds are pairwise distinct:

$$\mu_j \neq \mu_k \text{ and } \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \quad \forall j, k,\ j \neq k.$$

Proof sketch:

We may apply Dafermos criterion, or Benchimol or Arendt-Batty strong stability criterion. It suffices to show that $\mathcal{A}$ has no purely imaginary eigenvalue. One easily checks that $0 \in \rho(\mathcal{A})$. Now, let λ be a nonzero real number and let $Z = (u, v) \in D(\mathcal{A})$ with

$$\mathcal{A}Z = i\lambda Z. \qquad (*)$$

We shall show that $Z = (0, 0)$. It follows from $(*)$:

$$d(x)\sum_{j=1}^{q} u_j = 0 \text{ in } \Omega, \text{ and so, } -\lambda^2 u_j - \mu_j \Delta u_j - (\mu_j + \lambda_j)\nabla \mathrm{div} u_j = 0 \text{ in } \Omega.$$

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Therefore, setting $\varphi_j = \mathrm{div}(u_j)$ and $\ell_j = 1/(\lambda_j + 2\mu_j)$, it follows

$$\sum_{j=1}^{q} u_j = 0 \text{ in } \omega, \text{ and } -\lambda^2 \ell_j \varphi_j - \Delta\varphi_j = 0 \text{ in } \omega.$$

Using elementary algebra, one derives from the last two equations

$$\sum_{j=1}^{q} \ell_j^k \varphi_j = 0 \text{ in } \omega, \quad k = 0,\ 1,\ ...,\ q-1.$$

The determinant of that linear system is a Vandermonde determinant and is given by

$$D_q = \Pi_{1 \leq j < k \leq q}(\ell_k - \ell_j).$$

One checks that $D_q \neq 0$ if and only if $\lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k$ for all $j,\ k$ with $j \neq k$. In this case, $\varphi_j = 0$ in ω for each j.

$$D_q = \Pi_{1 \leq j < k \leq q}(\ell_k - \ell_j).$$

Consequently,

$$-\lambda^2 u_j - \mu_j \Delta u_j = 0 \text{ in } \omega.$$

Repeating the same arguments as above, we find, (setting $m_j = 1/\mu_j$):

$$\sum_{j=1}^{q} m_j^k u_j = 0 \text{ in } \omega, \quad k = 0,\ 1,\ ...,\ q-1.$$

As earlier, one derives $u_j = 0$ in ω for each j if and only if $\mu_j \neq \mu_k$ for all $j,\ k$ with $j \neq k$.

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The Imanuvilov-Yamamoto Carleman estimate for the static Lamé system [Appl. Anal. 2004] then yields $u_j = 0$ in Ω for each j. Hence $Z = (0, 0)$. □

A new unique continuation result

Theorem 3.

Let ω be an arbitrary nonvoid open set contained in Ω. Consider the uncoupled elastodynamic system

$$\begin{array}{l} y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(y_j) = 0 \text{ in } \Omega \times (0,\infty) \\ y_j = 0 \text{ on } \Gamma \times (0,\infty) \\ y_j(x,0) = y_j^0 \in [H_0^1(\Omega)]^N, \quad y_{jt}(.,0) = y_j^1 \in [L^2(\Omega)]^N, \\ j = 1,2,...,q. \end{array}$$

Assume that $\mu_j \neq \mu_k$ and $\lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \quad \forall j,k,\ j \neq k$, and there exists $T_0 > 0$ such that $\sum_{j=1}^{q} y_{kt} = 0$ in $\omega \times (0, T_0)$. Then

$$y_j = 0 \text{ in } \Omega \times (0, T_0), \quad \forall j.$$

Proof sketch:

Decompose the solution y of the uncoupled elastodynamic equations as: $y = w + z$ where w satisfies the damped system

$$
\begin{aligned}
& w_{jtt} - \mu_j \Delta w_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(w_j) + 1_\omega \sum_{k=1}^{q} w_{kt} = 0 \text{ in } \Omega \times (0,\infty) \\
& w_j = 0 \text{ on } \Gamma \times (0,\infty) \\
& w_j(x,0) = y_j^0(x), \quad w_{jt}(x,0) = y_j^1(x), \text{ in } \Omega, \\
& j = 1,\ 2, \ldots,\ q,
\end{aligned}
$$

and z is the solution of the system

$$
\begin{aligned}
& z_{jtt} - \mu_j \Delta z_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(z_j) = 1_\omega \sum_{k=1}^{q} w_{kt} \text{ in } \Omega \times (0,\infty) \\
& z_j = 0 \text{ on } \Gamma \times (0,\infty) \\
& z_j(x,0) = 0, \quad z_{jt}(x,0) = 0, \text{ in } \Omega, \\
& j = 1,\ 2, \ldots,\ q.
\end{aligned}
$$

Thanks to Theorem 2, we have

$$\lim_{t\to\infty} E_w(t) = 0.$$

On the other hand, the energy method shows

$$\begin{aligned} E_z(t) &= \int_0^t \int_\omega \sum_{k=1}^q p_{kt}(x,s) \sum_{j=1}^q z_{jt}(x,s)\, dxds \\ &= \int_0^t \int_\omega \sum_{k=1}^q y_{kt}(x,s) \sum_{j=1}^q z_{jt}(x,s)\, dxds \\ &\quad - \int_0^t \int_\omega \left| \sum_{k=1}^q z_{kt}(x,s) \right|^2 dxds. \end{aligned}$$

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So, if $\sum_{j=1}^q y_{kt} = 0$ in $\omega \times (0, T_0)$ for some $T_0 > 0$, then $E_z(t) = 0$ for all $t \in [0, T_0]$.

Consequently, $y = w$ on $\Omega \times (0, T_0)$. We know that for every $\varepsilon > 0$, there exists a time $T_\varepsilon > 0$, such that

$$t > T_\varepsilon \Rightarrow E_w(t) < \varepsilon.$$

Lamé systems with localized damping

Theorem4: Exponential stability

Let $(y_j^0, y_j^1)_j \in \left(\left[H_0^1(\Omega)\right]^N \times \left[L^2(\Omega)\right]^N\right)^q$. Suppose

$$\mu_j \neq \mu_k,\ \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k,\ \text{and}\ \lambda_j\mu_k = \lambda_k\mu_j, \quad \forall j, k,\ j \neq k.$$

Assume that ω satisfies the Liu geometric control condition, and suppose that the damping is effective in ω:

$$\exists d_0 > 0 : d(x) \geq d_0 \text{ a.e. } \omega.$$

There exist positive constants M and κ, independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t}E(0),\ \text{for all}\ t \geq 0.$$

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Proof method: FDM, multipliers technique, Huang or Prüss criterion.

An observability result

Let $T > 0$. Let ω be a nonempty open set in Ω satisfying the Liu geometric control condition. Consider the uncoupled elastodynamic system

$$
\begin{array}{l}
y_{jtt} - \mu_j \Delta y_j - (\mu_j + \lambda_j)\nabla \mathrm{div}(y_j) = 0 \text{ in } \Omega \times (0, T) \\
y_j = 0 \text{ on } \Gamma \times (0, T) \\
y_j(x,0) = y_j^0(x), \quad y_{jt}(x,0) = y_j^1(x), \text{ in } \Omega, \\
j = 1, \ 2, ..., \ q.
\end{array}
$$

There exists $T_0 > 0$ such that for any $T > T_0$, there exists $C > 0$:

$$
E(0) \leq \int_0^T \int_\omega |\sum_{j=1}^q y_{jt}(x,t)|^2 \, dxdt,
$$

provided that

$$
\mu_j \neq \mu_k, \ \lambda_j + 2\mu_j \neq \lambda_k + 2\mu_k, \text{ and } \lambda_j \mu_k = \lambda_k \mu_j, \quad \forall j, k, \ j \neq k.
$$

Euler-Bernoulli Plate-wave system

Consider the damped system

$$\begin{cases} y_{tt} - \Delta y + d(x)(y_t + z_t) = 0 \text{ in } \Omega \times (0, \infty) \\ z_{tt} + \Delta^2 z + d(x)(y_t + z_t) = 0 \text{ in } \Omega \times (0, \infty) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, \infty) \\ y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega), \\ z(0) = z^0 \in H_0^2(\Omega), \quad z_t(0) = z^1 \in L^2(\Omega). \end{cases}$$

The total energy is given, for all $t \geq 0$, by

$$2E(t) = \int_\Omega \{|y_t(x,t)|^2 + |\nabla y(x,t)|^2 + |z_t(x,t)|^2 + |\Delta z(x,t)|^2\}\, dx,$$

Euler-Bernoulli Plate-wave system

E is a nonincreasing function of the time variable as

$$\frac{dE}{dt} = -\int_{\Omega} d(x) \left| y_t(x,t) + z_t(x,t) \right|^2 \, dx.$$

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Question 1: Does the energy E decay to zero as time goes to infinity?

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Question 1: Does the energy E decay to zero as time goes to infinity?
Question 2: Under which conditions is the system exponentially stable?

Euler-Bernoulli Plate-wave system

Theorem 4: Strong stability

Let ω be an arbitrary nonvoid open set contained in Ω. Suppose that the damping coefficient d is positive in ω.
The system is strongly stable:

$$\lim_{t\to\infty} E(t) = 0$$

provided that either meas$(\partial\omega \cap \partial\Omega) > 0$, or else, the only solution of $\Delta u = -u$ in Ω and $u = 0$ on $\partial\Omega$ is $u = 0$.

Another new unique continuation result

Let $T > 0$. Let ω be an arbitrary nonvoid open set contained in Ω. Consider the uncoupled system

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } \Omega \times (0, T) \\ z_{tt} + \Delta^2 z = 0 \text{ in } \Omega \times (0, T) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, T). \end{cases}$$

There exists $T_0 > 0$ such that for any $T > T_0$,

$$y_t + z_t = 0 \text{ in } \omega \times (0, T) \Rightarrow y = 0 \text{ and } z = 0 \text{ in } \Omega \times (0, T),$$

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Let $T > 0$. Let ω be an arbitrary nonvoid open set contained in Ω. Consider the uncoupled system

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } \Omega \times (0, T) \\ z_{tt} + \Delta^2 z = 0 \text{ in } \Omega \times (0, T) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, T). \end{cases}$$

There exists $T_0 > 0$ such that for any $T > T_0$,

$$y_t + z_t = 0 \text{ in } \omega \times (0, T) \Rightarrow y = 0 \text{ and } z = 0 \text{ in } \Omega \times (0, T),$$

provided that meas$(\partial\omega \cap \partial\Omega) > 0$,or else, the only solution of $\Delta u = -u$ in Ω and $u = 0$ on $\partial\Omega$ is $u = 0$.

Euler-Bernoulli Plate-wave system

Theorem 5: Exponential stability

Let $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(z^0, z^1) \in H_0^2(\Omega) \times L^2(\Omega)$.
Assume that ω satisfies the Liu geometric control condition, and suppose that the damping is effective in ω:

$$\exists d_0 > 0 : d(x) \geq d_0 \text{ a.e. } \omega.$$

There exist positive constants M and κ, independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t} E(0), \text{ for all } t \geq 0.$$

An observability inequality

Let $T > 0$. Let ω be a nonempty open set in Ω satisfying the Liu geometric control condition.
Consider the uncoupled system

$$\begin{cases} y_{tt} - \Delta y = 0 \text{ in } \Omega \times (0, T) \\ z_{tt} + \Delta^2 z = 0 \text{ in } \Omega \times (0, T) \\ y = 0, \quad z = 0, \quad \partial_\nu z = 0 \text{ on } \Gamma \times (0, T). \end{cases}$$

There exists $T_0 > 0$ such that for any $T > T_0$, there exists $C > 0$:

$$E(0) \leq \int_0^T \int_\omega |y_t(x,t) + z_t(x,t)|^2 \, dxdt,$$

Timoshenko beam

Let $L > 0$, and set $\Omega = (0, L)$, and $\omega = (l_1, l_2)$ with $0 \leq l_1 < l_2 \leq L$. Consider the damped Timoshenko system:

$$\begin{cases} \rho_1 y_{tt} - k(y_x + z)_x + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty) \\ \rho_2 z_{tt} - \sigma z_{xx} + k(y_x + z) + a(x)(y_t + z_t) = 0 \text{ in } (0, L) \times (0, \infty), \end{cases}$$

with the boundary conditions:

(**DD**) $y(0, t) = 0, \quad y(L, t) = 0, \quad z(0, t) = 0, \quad z(L, t) = 0$, or else

(**DN**) $y(0, t) = 0, \quad y(L, t) = 0, \quad z_x(0, t) = 0, \quad z_x(L, t) = 0, \quad t > 0$

and the initial conditions:

$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$

Timoshenko beam

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$y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x), \quad z(x, 0) = z^0(x), \quad z_t(x, 0) = z^1(x),$

The damping coefficient a is a nonnegative bounded measurable function, which is positive in ω only.

The energy and main questions

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{\rho_1|y_t(x,t)|^2 + k|y_x(x,t) + z(x,t)|^2\}\,dx \\ +\tfrac{1}{2}\int_\Omega \{\rho_2|z_t(x,t)|^2 + \sigma|z_x(x,t)|^2\}\,dx, \quad \forall t \geq 0.$$

The energy E is a nonincreasing function of the time variable t as we have for every $t \geq 0$, (hereafter, $'$ denotes differentiation with respect to time)

$$E'(t) = -\int_\Omega a(x)|y_t(x,t) + z_t(x,t)|^2\,dx.$$

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$$E'(t) = -\int_\Omega a(x)|y_t(x,t) + z_t(x,t)|^2\,dx.$$

As before, our main purpose is to answer the following questions:

- Does the energy $E(t)$ decay to zero as the time variable t goes to infinity?
- If so, how fast? And if not, why?

Timoshenko beam

Theorem 6: Strong stability

Suppose that ω is an arbitrary nonempty open interval in Ω. Let the damping coefficient a be positive in ω. In either of the (**DD**) or (**DN**) case, the associated system is strongly stable:

$$\lim_{t\to\infty} E(t) = 0$$

if and only if $\partial\omega \cap \partial\Omega \neq \emptyset$.

Timoshenko beam

Theorem 7: Exponential stability

Suppose that ω is an arbitrary nonempty open interval in Ω with $\partial\omega \cap \partial\Omega \neq \emptyset$. Let the damping coefficient a satisfy

$$a(x) \geq a_0 > 0, \text{ a.e. in } \omega.$$

There exist positive constants M and κ, independent of the initial data, such that the following energy decay estimate holds:

$$E(t) \leq Me^{-\kappa t}E(0), \text{ for all } t \geq 0.$$

Brief literature

Notion explicitly introduced by Russell (1993), involves a coupled system of second order evolution equations where the damping occurs in one component of the system only.

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Notion explicitly introduced by Russell (1993), involves a coupled system of second order evolution equations where the damping occurs in one component of the system only.

We can broaden the notion to account for thermoelasticity or fluid-structure models where the dissipation is induced by the heat or parabolic component only.

Other contributors include Dafermos, Lasiecka and collaborators, Burns and collaborators, Lebeau-Zuazua, Perla Menzala-Zuazua, Rauch-Zhang-Zuazua, Triggiani-Avalos, Zhang-Zuazua, Alabau, Alabau-Cannarsa-Komornik,...

Mindlin-Timoshenko plate

$$
\begin{aligned}
&\rho_1 y_{tt} - k\mathrm{div}(\nabla y + z) = 0 \text{ in } \Omega \times (0, \infty) \\
&\rho_2 z_{tt} - \mu\Delta z - (\lambda + \mu)\nabla \mathrm{div} z + k(\nabla y + z) + a z_t = 0 \text{ in } \Omega \times (0, \infty) \\
&y = 0, \quad z = 0 \text{ on } \partial\Omega \times (0, \infty) \\
&y(.,0) = y^0 \in H_0^1(\Omega), \quad y_t(.,0) = y^1 \in L^2(\Omega), \\
&z(.,0) = z^0 \in [H_0^1(\Omega)]^N, \quad z_t(.,0) = z^1 \in [L^2(\Omega)]^N.
\end{aligned}
$$

Mindlin-Timoshenko plate

$$
\begin{aligned}
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&y = 0, \quad z = 0 \text{ on } \partial\Omega \times (0,\infty)\\
&y(.,0) = y^0 \in H_0^1(\Omega), \quad y_t(.,0) = y^1 \in L^2(\Omega),\\
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\end{aligned}
$$

In the one-dimensional setting, the system , known as the Timoshenko beam equations, describes the motion of a beam when the effects of rotatory inertia are accounted for; the transverse displacement is represented by y while z denotes the shear angle displacement.

In 2D, that system is known as the Mindlin-Timoshenko plate equations, where y represents the vertical deflection and z stands for the rotation angles of a filament.

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The constants ρ_1, ρ_2, k, and μ are physical constants and are all positive. In particular, the constants λ and μ are the Lamé constants with $\lambda + \mu > 0$.

Mindlin-Timoshenko plate

2009: Fernández-Sare shows that the system is polynomially stable.

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It is well-known that the indirectly damped Timoshenko beam, ($N = 1$), is exponentially stable if and only if

$$(*) \qquad \frac{k}{\rho_1} = \frac{2\mu + \lambda}{\rho_2}.$$

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$$(*) \qquad \frac{k}{\rho_1} = \frac{2\mu + \lambda}{\rho_2}.$$

Questions: Is the Mindlin-Timoshenko system exponentially stable under $(*)$? What happens when $(*)$ fails?

Energy estimates

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{\rho_1|y_t(x,t)|^2 + k|\nabla y(x,t) + z(x,t)|^2\}\,dx$$
$$+\frac{1}{2}\int_\Omega \{\rho_2|z_t(x,t)|^2 + \mu|\nabla z(x,t)|^2 + (\lambda+\mu)|\text{div} z(x,t)|^2\}\,dx, \quad \forall t \geq$$

Let ω satisfy Liu geometric constraint. Suppose that the damping coefficient a further satisfies

$$\exists a_0 > 0 : a(x) \geq a_0, \text{ a.e. } \omega.$$

Energy estimates

- If $(*)$ holds, then the energy decays exponentially:

$$\exists M > 0,\ \exists \zeta > 0 : E(t) \leq Me^{-\zeta t}E(0), \quad \forall t \geq 0.$$

- If $(*)$ fails, then the energy decays polynomially:

$$\exists M = M(\text{initial data}) > 0,\ \exists \zeta > 0 : E(t) \leq \frac{M}{(1+t)},$$

provided

$$(y^0, y^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$$

and

$$(z^0, z^1) \in [(H^2(\Omega) \cap H_0^1(\Omega))]^N \times [H_0^1(\Omega)]^N.$$

Kirchhoff plate-wave

Joint work with Ahmed Hajej (U. Cergy-Pontoise, France) and Zayd Hajjej (U. Gabes, Tunisia)

Undamped Kirchhoff plate/ damped wave

Consider the following weakly coupled system of Kirchhoff plate and wave equations:

$$
\begin{cases}
u_{tt} - \gamma\Delta u_{tt} + \Delta^2 u + \alpha v = 0 & \text{in} \quad \Omega \times (0,\infty) \\
v_{tt} - \Delta v + v_t + \alpha u = 0 & \text{in} \quad \Omega \times (0,\infty) \\
u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\
\Delta u + (1-\mu) B_1 u = 0 & \text{on} \quad \Gamma_0 \times (0,\infty) \\
\partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1-\mu) B_2 u = 0 & \text{on} \quad \Gamma_1 \times (0,\infty) \\
v = 0 & \text{on} \quad \Gamma \times (0,\infty) \\
u(0) = u^0 \in V, \ \ u_t(0) = u^1 \in H_0^1(\Omega), & \\
v(0) = v^0 \in H_0^1(\Omega), \ \ v_t(0) = v^1 \in L^2(\Omega). &
\end{cases}
$$

Undamped Kirchhoff plate/ damped wave

Ω is an open set of $\mathbb{R}^2$ with regular boundary $\Gamma = \partial\Omega = \Gamma_0 \cup \Gamma_1$ such that $\overline{\Gamma}_0 \cap \overline{\Gamma}_1 = \emptyset$,
The constant $\gamma > 0$ is the rotational inertia of the plate and the constant $0 < \mu < \frac{1}{2}$ is the Poisson coefficient.
The boundary operators B_1, B_2 are defined by

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx},$$

$$B_2 u = \partial_\tau \left((\nu_1^2 - \nu_2^2) u_{xy} + \nu_1\nu_2 (u_{yy} - u_{xx}) \right),$$

where $\nu = (\nu_1, \nu_2)$ is the unit outer normal vector to Γ and $\tau = (-\nu_2, \nu_1)$ is a unit tangent vector.

Energy estimates.

Introduce the energy, (setting $P_\gamma u = u - \gamma \Delta u$)

$$E(t) = \frac{1}{2} \int_\Omega \{ |P_\gamma^{\frac{1}{2}} u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv \}(x,t)\, dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{3}}} \left(||u^0||^2_{H^3(\Omega)} + ||u^1||^2_{H^2(\Omega)} + ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H^1_0(\Omega)} \right).$$

FDM, interpolation, good choice of functional inequalities, Borichev-Tomilov criterion.

Damped Kirchhoff plate/ undamped wave

$$\begin{cases} u_{tt} - \gamma \Delta u_{tt} + \Delta^2 u + \alpha v + u_t = 0 & \text{in} \quad \Omega \times (0, \infty) \\ v_{tt} - \Delta v + \alpha u = 0 & \text{in} \quad \Omega \times (0, \infty) \\ u = \partial_\nu u = 0 & \text{on} \quad \Gamma_0 \times (0, \infty) \\ \Delta u + (1 - \mu) B_1 u = 0 & \text{on} \quad \Gamma_0 \times (0, \infty) \\ \partial_\nu \Delta u - \gamma \partial_\nu u_{tt} + (1 - \mu) B_2 u = 0 & \text{on} \quad \Gamma_1 \times (0, \infty) \\ v = 0 & \text{on} \quad \Gamma \times (0, \infty) \\ u(0) = u^0 \in V, \quad u_t(0) = u^1 \in H_0^1(\Omega), & \\ v(0) = v^0 \in H_0^1(\Omega), \quad v_t(0) = v^1 \in L^2(\Omega). & \end{cases}$$

Energy estimates.

Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{|p_\gamma u_t|^2 + |\Delta u|^2 + |v_t|^2 + |\nabla v|^2 + 2\alpha uv\}(x,t)\,dx.$$

We have:

$$E(t) \leq \frac{C_\alpha}{(t+1)^{\frac{1}{4}}}\left(||u^0||^2_{H^3(\Omega)} + ||u^1||^2_{H^2(\Omega)} + ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H^1_0(\Omega)}\right).$$

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And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!