MAP 2302 (Differential Equations) — Answers
QUIZ 7, Friday March 23, 2018

Name: **PID:**

We have: $\mathcal{L}(t^n)(s) = n!/s^{n+1}$, $\mathcal{L}(\sin(bt))(s) = b/(s^2+b^2)$, $\mathcal{L}(\cos(bt))(s) = s/(s^2+b^2))$, $\mathcal{L}(e^{at})(s) = 1/(s-a)$.

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1. [4] If f is the solution of the initial-value problem:
$3y'' - 7y' + 4y = e^{-3t}\cos(5t)$, $\quad y(0) = 4$, $\quad y'(0) = -1$, find the Laplace transform F of f.

$\mathcal{L}(3y''-7y'+4y)(s) = \mathcal{L}(e^{-3t}\cos(5t))(s)$

$\mathcal{L}(y'')(s) = s^2F(s) - sy(0) - y'(0)$, $\mathcal{L}(y')(s) = sF(s) - y(0) = sF(s) - 4$
$= s^2F(s) - 4s + 1$ $\quad \mathcal{L}(e^{-3t}\cos 5t)(s) = \dfrac{s+3}{(s+3)^2+25}$, by the translation property

So D-E becomes:

$(3s^2-7s+4)F(s) - 12s + 3 + 28 = \dfrac{s+3}{(s+3)^2+25}$

So $\quad F(s) = \dfrac{12s-31}{3s^2-7s+4} + \dfrac{s+3}{((s+3)^2+25)(3s^2-7s+4)}$

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2. [4] Use properties of the Laplace transform to find: (Show all your work)

a) $\mathcal{L}(t\sin(4t))(s) = -\dfrac{d}{ds}\mathcal{L}(\sin(4t))(s)$, by monomial factor property
$= -\dfrac{d}{ds}\left(\dfrac{4}{s^2+16}\right) = -\dfrac{(-8s)}{(s^2+16)^2} = \dfrac{8s}{(s^2+16)^2}$

b) $\mathcal{L}(t^3e^{-5t})(s) = \dfrac{6}{(s+5)^4}$, by translation property (method 1)

$\mathcal{L}(t^3e^{-5t})(s) = (-1)^3\dfrac{d^3}{ds^3}\mathcal{L}(e^{-5t})(s) = -\dfrac{d^3}{ds^3}\left(\dfrac{1}{s+5}\right) = -\dfrac{d^2}{ds^2}\left(\dfrac{-1}{(s+5)^2}\right) = -\dfrac{d}{ds}\left(\dfrac{2}{(s+5)^3}\right)$
$= -\dfrac{d}{ds}\left(\dfrac{2}{(s+5)^3}\right) = -\left(\dfrac{-6}{(s+5)^4}\right) = \dfrac{6}{(s+5)^4}$, by monomial factor property

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3. [2] Use the definition of the Laplace transform to find $\mathcal{L}(te^{7t})(s)$.

$\mathcal{L}(te^{7t})(s) = \int_0^\infty t e^{(7-s)t}\,dt$
$= \lim_{A\to\infty}\int_0^A t e^{(7-s)t}\,dt$, $\quad s > 7$
$= \lim_{A\to\infty}\left[t\dfrac{e^{(7-s)t}}{7-s}\Big]_0^A - \int_0^A \dfrac{e^{(7-s)t}}{7-s}\,dt\right]$
$= \lim_{A\to\infty}\left[\dfrac{Ae^{(7-s)A}}{7-s} - \dfrac{e^{(7-s)t}}{(7-s)^2}\Big]_0^A\right]$
$= \lim_{A\to\infty}\left[\dfrac{Ae^{(7-s)A}}{7-s} - \dfrac{e^{(7-s)A}}{(7-s)^2} + \dfrac{1}{(s-7)^2}\right]$
$= \dfrac{1}{(7-s)^2} = \dfrac{1}{(s-7)^2}$, as $\lim_{A\to\infty} Ae^{(7-s)A} = 0 = \lim_{A\to\infty} e^{(7-s)A}$, since $s > 7$.

<u>Note</u>: Improper integral diverges for $s \le 7$.