MAP 2302 (Differential Equations) — Answers
QUIZ 8, Friday March 30, 2018

Name: **PID:**

We have: $\mathcal{L}(t^n)(s) = n!/s^{n+1}$, $\mathcal{L}(\sin(bt))(s) = b/(s^2+b^2)$, $\mathcal{L}(\cos(bt))(s) = s/(s^2+b^2))$, $\mathcal{L}(e^{at})(s) = 1/(s-a)$.

1. [6] Use Laplace transform to solve the initial-value problem:
$y'' + 2y' + y = te^{-t}$, $\quad y(0) = 1$, $\quad y'(0) = -1$. (Show all your work) Set $Y(s) = \mathcal{L}(y)(s)$

$\mathcal{L}(y''+2y'+y)(s) = \mathcal{L}(te^{-t})(s) = \frac{1}{(s+1)^2}$, by the translation property

$\mathcal{L}(y'')(s) = s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s + 1$

$\mathcal{L}(y')(s) = sY(s) - y(0) = sY(s) - 1$

D.E becomes then:

$(s^2+2s+1)Y(s) = s+1 + \frac{1}{(s+1)^2}$ or $(s+1)^2Y(s) = s+1 + \frac{1}{(s+1)^2}$

So $Y(s) = \frac{s+1}{(s+1)^2} + \frac{1}{(s+1)^4} = \frac{1}{s+1} + \frac{1}{(s+1)^4}$

Hence $y(t) = \mathcal{L}^{-1}(Y)(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)(t) + \frac{1}{6}\mathcal{L}^{-1}\left(\frac{6}{(s+1)^4}\right)(t)$
$= e^{-t} + \frac{1}{6}t^3e^{-t}$, by the translation property

2. [2] Find the inverse Laplace transform: (Show all your work)
$\mathcal{L}^{-1}\left(\frac{2s}{(s^2+4)^2}\right)(t) = \frac{1}{2}t\sin(2t)$

Note: $\frac{d}{ds}\frac{2}{s^2+4} = \frac{-4s}{(s^2+4)^2}$; so $(-1)^1\frac{d}{ds}\left(\frac{2}{s+4}\right) = \frac{4s}{(s^2+4)^2} = \mathcal{L}(t\sin(2t))(s)$, by the monomial factor property

3. [2] Use the convolution to find the inverse Laplace transform: (Show all your work)

$\mathcal{L}^{-1}\left(\frac{3}{s^2-s-6}\right)(t) = \mathcal{L}^{-1}\left(\frac{3}{(s-3)(s+2)}\right)(t)$

$= 3\mathcal{L}^{-1}\left(\frac{1}{s-3}\cdot\frac{1}{s+2}\right)(t)$; $\quad \mathcal{L}^{-1}\left(\frac{1}{s-3}\right)(t) = e^{3t}$, $\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)(t) = e^{-2t}$

$= 3(e^{3t} * e^{-2t})(t)$

$= 3\int_0^t e^{3(t-r)}e^{-2r}\,dr = 3e^{3t}\int_0^t e^{-5r}\,dr$

$= 3e^{3t}\left[-\frac{e^{-5r}}{5}\right]_0^t$

$= \frac{3}{5}e^{3t}(1-e^{-5t})$