Controllability of some coupled hyperbolic systems

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Overview

- Hyperbolic equations with internal coupling.

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- Hyperbolic equations with boundary coupling.

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- Hyperbolic equations with boundary coupling.
- Some open problems.

Notations

Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
Γ= boundary of Ω is smooth,
$T > 0$, $Q = \Omega \times (0, T)$
$\omega =$ nonvoid open subset in Ω.
The coefficients matrix $(b_{ij})_{i,j}$, satisfies:

$$b_{ij} \in C^1(\bar{\Omega}); \quad b_{ij} = b_{ji}, \quad \forall i,\ j = 1, 2, ..., N,$$
$$\exists a_0 > 0 : b_{ij}(x) z_i z_j \geq a_0 z_i z_i, \quad \forall (x, z) \in \bar{\Omega} \times \mathbb{R}^N.$$

The Einstein summation convention on repeated indices is used throughout.
a, b, c, d lie in $L^\infty(0, T; L^s(\Omega))$, $s \geq \max(2, N)$ for $N \neq 2$,
and $s > 2$ for $N = 2$.
k_{ij}, l_{ij} lie in $W_0^{1,\infty}(0, T; L^s(\Omega))$.

Controllability

Consider the controllability problems: Given (z^0, z^1) and (w^0, w^1), and $\varepsilon > 0$, find a control h such that if (z, w) solves the system

$$\begin{cases} z_{tt} - \partial_i(b_{ij}(x)\partial_j z) + az + cw - \mathrm{div}(k_{11} z) - (l_{11} z)_t \\ -\mathrm{div}(k_{21} w) - (l_{21} w)_t = h1_\omega \text{ in } Q \\ \\ w_{tt} - \partial_i(b_{ij}(x)\partial_j w) + bz + dw - \mathrm{div}(k_{12} z) - (l_{12} z)_t \\ -\mathrm{div}(k_{22} w) - (l_{22} w)_t = 0 \text{ in } Q \\ \\ z = 0, \quad w = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ \\ z(0) = z^0; \quad z_t(0) = z^1 \quad w(0) = w^0; \quad w_t(0) = w^1 \text{ in } \Omega, \end{cases}$$

then (exact controllability)

$$z(T) = 0, \quad z_t(T) = 0, \quad w(T) = 0, \quad w_t(T) = 0 \text{ in } \Omega,$$

or else (approximate controllability)

$$||z(T)||_1 + ||z_t(T)||_2 \leq \varepsilon, \quad ||w(T)||_1 + ||w_t(T)||_2 \leq \varepsilon.$$

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- For approximate controllability, only T must be large enough.
- Lions' HUM reduces exact controllability to an inverse (observability) estimate for the adjoint system.

Observability estimates

Consider the coupled (adjoint) system

$$\begin{cases} u_{tt} - \partial_i(b_{ij}(x)\partial_j u) + au + bv + k_{11} \cdot \nabla u + l_{11} u_t \\ +k_{12} \cdot \nabla v + l_{12} v_t = 0 \text{ in } Q \\ \\ v_{tt} - \partial_i(b_{ij}(x)\partial_j v) + cu + dv + k_{21} \cdot \nabla u + l_{21} u_t \\ +k_{22} \cdot \nabla v + l_{22} v_t = 0 \text{ in } Q \\ \\ u = 0, \quad v = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ \\ u(0) = u^0; \quad u_t(0) = u^1 \quad v(0) = v^0; \quad v_t(0) = v^1 \text{ in } \Omega. \end{cases}$$

The coupled system is well-posed in $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$.

Introduce the energies:

$$E_u(t) = \frac{1}{2}\int_\Omega \{|u_t(x,t)|^2 + (b_{ij}(x)\partial_j u(x,t)\partial_i u(x,t)\}\, dx,$$

$$\widehat{E}_u(t) = \frac{1}{2}\left(||u(.,t)||^2_{L^2(\Omega)} + ||u_t(.,t)||^2_{H^{-1}(\Omega)}\right).$$

For each $t \in [0, T]$, set

$$E(t) = E_u(t) + \widehat{E}_v(t), \quad \widehat{E}(t) = \widehat{E}_u(t) + \widehat{E}_v(t).$$

Introduce a function $h \in C^2(\bar{\Omega})$ satisfying for some $m_0 \geq 4$:

i) $\left(2b_{il}(b_{kj}h_{x_k})_{x_l} - b_{ij,x_l}b_{kl}h_{x_k}\right) z_i z_j \geq m_0 b_{ij} z_i z_j, \quad \forall (x,z) \in \bar{\Omega} \times \mathbb{R}^N.$
ii) $\min\left\{|\nabla h(x)|; x \in \bar{\Omega}\right\} > 0.$
iii) $\frac{1}{4} b_{ij}(x) h_{x_i}(x) h_{x_j}(x) \geq R_1^2 \geq R_0^2 > 0, \quad \forall x \in \bar{\Omega},$

where $R_0 = \min\left\{\sqrt{h(x)}; x \in \bar{\Omega}\right\}$, and $R_1 = \max\left\{\sqrt{h(x)}; x \in \bar{\Omega}\right\}$. Let ν be the unit normal pointing into the exterior of Ω, and set

$$\Gamma_0 = \left\{x \in \partial\Omega; b_{ij}\nu_i h_{x_j}(x) > 0\right\}.$$

Theorem 1

Let ω and $\mathcal{O}$ be neighborhoods of Γ_0. Assume that $a,\ c,\ d \in L^\infty(0, T; L^s(\Omega))$, with $s > 2$ for $N \in \{1,\ 2\}$ and $s \geq N$ for $N \geq 3$. Let $b \in L^\infty(Q)$, and let $k_{ij} \in (W_0^{1,s}(Q) \cap L^\infty(Q))^N$, $l_{ij} \in W_0^{1,s}(Q) \cap L^\infty(Q)$, $i, j = 1, 2$. Suppose that $k_{12} \equiv 0$, $l_{12} \equiv 0$, $\mathrm{supp}(k_{22}) \subset \omega_0 \times (0, T)$, and $\mathrm{supp}(l_{22}) \subset \omega_0 \times (0, T)$, where ω_0 is another neighborhood of Γ_0 whose closure $\bar{\omega}_0$ is contained in $\mathcal{O} \cap \omega$. Suppose that there exists $b_0 > 0$ such that $b(x, t) \geq b_0$ for almost every (x, t) in $\mathcal{O} \times (0, T)$.

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For every $T > 2R_1$, there exists a positive constant C such that for all $(u^0,u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(v^0,v^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, one has the observability estimate:

$$E(0) \leq C \int_0^T \int_\omega (|u_t|^2 + |u|^2)\, dxdt$$

for the corresponding solution pair (u,v) of the adjoint system.

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4. The support constraints on k_{22} and l_{22} are used in the proof of the observability estimate to absorb some unwanted terms, but they may be replaced with smallness constraints instead.

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4. The support constraints on k_{22} and l_{22} are used in the proof of the observability estimate to absorb some unwanted terms, but they may be replaced with smallness constraints instead.
5. One may fairly wonder whether the observability estimate in Theorem 1 may be replaced with

$$E(0) \leq C \int_0^T \int_\omega |u_t|^2 \, dxdt.$$

But as noted in the case of a single wave equation, that estimate is false in general, but holds under some constraints on the potential.

Some literature

- Dáger (2006), $\Omega = (0, 1)$, $T \geq 4$, $b = -1_{\mathcal{O}}$, all other *l.o.t* vanish. Proved a weaker estimate; see Theorem 2 in the sequel.

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  (GCC) [Bardos-Lebeau-Rauch, 1988, 1992]: every ray of geometric optics enters ω, (resp. $\mathcal{O}$) in a time less than T.
  But the controllability time blows up as the norm of the coupling function b goes to zero; this is not natural. One would expect the controllability cost to blow up as the coupling goes to zero, but not the controllability time.

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- Tebou (2012), nonconservative systems.

Proof of Theorem 1: key elements

- Energy estimates show

$$E(0) \leq C \int_{Q_0} \{|u_t|^2 + |\nabla u|^2 + |v|^2\} \, dxdt,$$

where Q_0 is an appropriate subset of Q.

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- Fu-Yong-Zhang Carleman estimate shows

$$\int_{Q_0} (|u_t|^2 + |\nabla u|^2 + |v|^2)\, dxdt \leq Ce^{-\mu\lambda} E(0) + C \int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt \\ + C \int_0^T \int_{\omega} (|u_t|^2 + |u|^2)\, dxdt$$

where $\lambda > 0$ is large enough, and $\mu > 0$ is fixed.

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where $\lambda > 0$ is large enough, and $\mu > 0$ is fixed.

- Use a localizing argument to absorb $C \int_0^T r^2 \int_{\omega_0} |v|^2\, dxdt$.

Set

$$\delta = ||a||_{\infty,s} + ||b||_{\infty,s} + ||c||_{\infty,s} + ||d||_{\infty,s} + \sum_{i,j=1}^{2} ||\mathrm{div}(k_{ij})||_{\infty,s} + \sum_{i,j=1}^{2} ||l_{ij,t}||_{\infty,s}$$

$$\delta_0 = \sum_{i,j=1}^{2} ||k_{ij}||_{\infty} + \sum_{i,j=1}^{2} ||l_{ij}||_{\infty}$$

where $||.||_{\infty,s} = ||.||_{L^\infty(0,T;L^s(\Omega))}$, and $||.||_{\infty} = ||.||_{L^\infty(Q)}$.

Theorem 2.

Let ω, $\mathcal{O}$, a, d and s be as in Theorem 1, and suppose that $b \in L^\infty(0, T; L^s(\Omega))$, $c \in L^\infty(Q)$, and there exists $b_0 > 0$ such that $b(x,t) \geq b_0$ for almost every (x,t) in $\mathcal{O} \times (0, T)$. Let $k_{ij} \in (W_0^{1,s}(Q) \cap L^\infty(Q))^N$, $l_{ij} \in W_0^{1,s}(Q) \cap L^\infty(Q)$, $i, j = 1, 2$. Suppose that $k_{21} \equiv 0$, $l_{21} \equiv 0$, $\text{supp}(k_{ij}) \subset \omega_0 \times (0, T)$, and $\text{supp}(l_{ij}) \subset \omega_0 \times (0, T)$.

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For every $T > 2R_1$, there exists a positive constant $C_0 = C_0(\Omega, \omega, \mathcal{O}, T, N, s)$ such that for all $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, and $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, one has the observability estimate:

$$\widehat{E}(0)^2 \leq e^{C_0(1+\delta_0+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |u|^2 \, dxdt \right) (\widehat{E}_u(0) + E_v(0))$$

for all solution pair (u, v) of the adjoint system.

Proof of Theorem 2: Main ideas

Step 1. Prove the energy estimates

- $$\widehat{E}(t) \leq \left[\exp C_0(1 + \delta_0 + \delta^{\frac{N+s}{2s}})|t - \tau|\right] \widehat{E}(\tau), \qquad \forall \tau, t \in [0, T],$$

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- $$\int_{T_0}^{T_0'} h\widehat{E}(t)\,dt \leq C_0(1+\delta+\delta_0)\int_{Q_0}\{|u|^2+|v|^2\}\,dxdt,$$

where h is an appropriate cut-off function.

Step 2. Derive from Step 1

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Step 3. Duyckaerts-Zhang-Zuazua Carleman estimate yields

$$\int_{Q_0} (|u|^2 + |v|^2) \, dxdt \leq e^{-C_0\lambda} \widehat{E}(0) + e^{C_0\lambda} \int_0^T \int_\omega |u|^2 \, dxdt + e^{C_0\lambda} \int_0^T r^2 \int_{\omega_0} |v|^2 \, dxdt,$$

for some constants $C_0 = C_0(\Omega, T, N, s, \omega) > 0$, and for all $\lambda \geq C_0(1 + \delta_0 + \delta^{\frac{2s}{3s-2N}})$.

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for some constants $C_0 = C_0(\Omega, T, N, s, \omega) > 0$, and for all $\lambda \geq C_0(1 + \delta_0 + \delta^{\frac{2s}{3s-2N}})$.

Step 4. Use a localizing argument to absorb $e^{C_0\lambda} \int_0^T r^2 \int_{\omega_0} |v|^2 \, dxdt$.

Let $a,\ b,\ c,\ d \in L^s(\Omega)$, with s as in Theorem 1. Assume now $l_{ij} \equiv 0$, and $k_{ij} \equiv 0$, $i, j = 1, 2$. Let ω, $\mathcal{O}$, be as in Theorem 1, and suppose that there exists $b_0 > 0$ such that $b(x) \geq b_0$ for almost every x in $\mathcal{O}$.
Further assume that either:

$$a \geq 0, \quad d \geq 0, \quad 2a - |b + c| \geq 0, \text{ and } 2d - |b + c| \geq 0, \text{ a.e. } x \in \Omega$$

or else

$$a \geq 0, \quad d \geq 0, \text{ a.e. } x \in \Omega, \quad 1 - C_s^2 |b + c|_s > 0, \text{ and } \lambda_0^2 - |b + c|_s > 0,$$

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where λ_0^2 is the first eigenvalue of the operator $-\partial_i(b_{ij}(x)\partial_j)$ under Dirichlet boundary conditions, and C_s denotes the best constant in the Sobolev inequality:

$$||w||^2_{\frac{2s}{s-2}} \leq C_s^2 \int_\Omega b_{ij}(x)\partial_j w(x)\partial_i w(x)\, dx, \quad \forall w \in H_0^1(\Omega).$$

Theorem 3

Assume the hypotheses just stated. For every $T > 2R_1$, there exists a positive constant $C_0 = C_0(\Omega, \omega, \mathcal{O}, T, N, s)$ such that for all $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $(v^0, v^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, one has the observability estimate:

$$(E_u(0) + E_v(0))^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |u_t|^2 \, dxdt \right) (E_u(0) + \check{E}_v(0))$$

for all solution pair (u, v) of the adjoint system, and where $2\check{E}_v(0) = ||v^0||^2_{H^2(\Omega)} + ||v^1||^2_{H_0^1(\Omega)}$.

Sketch of the proof of Theorem 3

For this proof, we shall use Theorem 2, and the following result

Lemma

Let $a,\ b,\ c$, and d be given as in Theorem 3. Then there exists a positive constant $C_0 = C_0(\Omega, b+c)$ such that

$$||-\partial_i(b_{ij}(x)\partial_j u) + au + bv||^2_{H^{-1}(\Omega)} + ||-\partial_i(b_{ij}(x)\partial_j v) + cu + dv||^2_{H^{-1}(\Omega)} \\ \geq C_0 \int_\Omega \{b_{ij}(x)\partial_j u \partial_i u + b_{ij}(x)\partial_j v \partial_i v\}\, dx, \quad \forall u, v \in H^1_0(\Omega).$$

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$$\geq C_0 \int_\Omega \{b_{ij}(x)\partial_j u \partial_i u + b_{ij}(x)\partial_j v \partial_i v\}\, dx, \quad \forall u, v \in H^1_0(\Omega).$$

Set $\hat{w} = u_t$ and $\hat{z} = v_t$. Then these functions solve the system

$$\begin{cases} \hat{w}_{tt} - \partial_i(b_{ij}(x)\partial_j \hat{w}) + a\hat{w} + b\hat{z} = 0 \text{ in } Q \\ \hat{z}_{tt} - \partial_i(b_{ij}(x)\partial_j \hat{z}) + c\hat{w} + d\hat{z} = 0 \text{ in } Q \\ \hat{w} = 0, \quad \hat{z} = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ \hat{w}(0) = u^1 \in L^2(\Omega); \quad \hat{w}_t(0) = \partial_i(b_{ij}(x)\partial_j u^0) - au^0 - bv^0 \in H^{-1}(\Omega) \\ \hat{z}(0) = v^1 \in H^1_0(\Omega); \quad \hat{z}_t(0) = \partial_i(b_{ij}(x)\partial_j v^0) - cu^0 - dv^0 \in L^2(\Omega). \end{cases}$$

Introduce the following energy associated with that system

$$\widehat{E}_{\hat{w},\hat{z}}(t) = \widehat{E}_{\hat{w}}(t) + \widehat{E}_{\hat{z}}(t) \quad \forall t \in [0, T].$$

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Thanks to Theorem 2, one has:

$$\widehat{E}_{\hat{w},\hat{z}}(0)^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |\hat{w}|^2 \, dxdt \right) (\widehat{E}_{\hat{w}}(0) + E_{\hat{z}}(0)).$$

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Some elementary calculations show that

$$\widehat{E}_{\hat{w}}(0) + E_{\hat{z}}(0) \leq C_0(E_u(0) + \check{E}_v(0)),$$

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$$\widehat{E}_{\hat{w},\hat{z}}(t) = \widehat{E}_{\hat{w}}(t) + \widehat{E}_{\hat{z}}(t) \quad \forall t \in [0, T].$$

Thanks to Theorem 2, one has:

$$\widehat{E}_{\hat{w},\hat{z}}(0)^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |\hat{w}|^2 \, dxdt \right) (\widehat{E}_{\hat{w}}(0) + E_{\hat{z}}(0)).$$

Some elementary calculations show that

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Hence

$$(E_u(0) + E_v(0))^2 \leq e^{C_0(1+\delta^{\frac{2s}{3s-2N}})} \left(\int_0^T \int_\omega |u_t|^2 \, dxdt \right) (E_u(0) + \check{E}_v(0)).$$

Theorem 4

Suppose that the hypotheses of Theorem 3 hold. For every $T > 2R_1$, there exists a positive constant $C = C(\Omega, \omega, \mathcal{O}, T, N, s, a, b, c, d)$ such that for all $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, and $(v^0, v^1) \in H_0^1(\Omega) \times L^2(\Omega)$, one has the observability estimate:

$$\check{E}_u(0) + E_v(0) \leq C \int_0^T \int_\omega \{|u_t|^2 + |u_{tt}|^2\} \, dxdt.$$

Controllability

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable function with

$$\limsup_{|s|\to\infty} \frac{|f(s)|}{|s|(\log|s|)^{\alpha}} = \beta_0,$$

for some $\beta_0 > 0$, and some $0 \leq \alpha < 3/2$. Consider now the controllability problem: Given $y^0,\ \tilde{y}^0 \in H_0^1(\Omega)$, and $y^1,\ \tilde{y}^1 \in L^2(\Omega)$; $q^0,\ \tilde{q}^0 \in L^2(\Omega)$, and $q^1,\ \tilde{q}^1 \in H^{-1}(\Omega)$; and $\xi \in L^2(Q)$, can we find a control $v \in L^2(0,T;L^2(\omega))$ such that the corresponding solution pair (y_0, q) of the cascade system:

$$\begin{cases} y_{0tt} - \partial_i(b_{ij}(x)\partial_j y_0) + f(y_0) = \xi + v\chi_\omega \text{ in } Q \\ q_{tt} - \partial_i(b_{ij}(x)\partial_j q) + f'(y_0)q = 0 \text{ in } Q \\ y_0 = 0, \quad q = \frac{\partial y_0}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1, \quad q(0) = q^0; \quad q_t(0) = q^1 \text{ in } \Omega, \end{cases}$$

satisfies:

$$y_0(.,T) = \tilde{y}^0, \quad y_{0t}(.,T) = \tilde{y}^1, \quad q(.,T) = \tilde{q}^0, \quad q_t(.,T) = \tilde{q}^1 \text{ in } \Omega?$$

For this system we have the controllability result:

Theorem 5

Assume that ω is a neighborhood of Γ_0. For every $T > 2R_1$, and for all $y^0 \in H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$, $q^0 \in L^2(\Omega)$ and $q^1 \in H^{-1}(\Omega)$, there exists a control $v \in L^2(0, T; L^2(\omega))$ such that

$$y_0(., T) = \tilde{y}^0, \quad y_{0t}(., T) = \tilde{y}^1, \quad q(., T) = \tilde{q}^0, \quad q_t(., T) = \tilde{q}^1 \text{ in } \Omega.$$

To prove Theorem 5, we're going to follow the classical algorithm for solving control problems for nonlinear distributed systems:

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To prove Theorem 5, we're going to follow the classical algorithm for solving control problems for nonlinear distributed systems:

1. linearize the control problem,
2. solve the linear control problem,
3. use a fixed-point theorem to derive the controllability of the nonlinear problem from that of the linearized system.

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- Duyckaerts-Zhang-Zuazua (2008), improved Carleman estimates, allows $\limsup_{|s|\to\infty} \dfrac{f(s)}{s(\log|s|)^{\alpha}} = 0, \quad 0 \leq \alpha < 3/2.$

The linear controllability problem

Set

$$g(s) = \begin{cases} (f(s) - f(0))/s, \text{ if } s \neq 0 \\ f'(0), \text{ if } s = 0. \end{cases}$$

Let $w \in L^\infty(0, T; L^2(\Omega))$. Set
$a(x, t) = g(w(x, t)), \quad b(x, t) = f'(w(x, t))$. The nonlinear controlled cascade system may be linearized as:

$$\begin{cases} y_{0tt} - \partial_i(b_{ij}(x)\partial_j y_0) + a(x, t)y_0 = -f(0) + \xi + v\chi_\omega \text{ in } Q \\ q_{tt} - \partial_i(b_{ij}(x)\partial_j q) + b(x, t)q = 0 \text{ in } Q \\ y_0 = 0, \quad q = \frac{\partial y_0}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1; \quad q(0) = q^0; \quad q_t(0) = q^1 \text{ in } \Omega \end{cases}$$

We shall find a control v so that:

$$y(T) = \tilde{y}^0; \quad y_t(T) = \tilde{y}^1, \quad q(T) = \tilde{q}^0; \quad q_t(T) = \tilde{q}^1 \text{ in } \Omega.$$

To this end, introduce the adjoint system:

$$\begin{cases} p_{tt} - \partial_i(b_{ij}(x)\partial_j p) + b(x,t)p = 0 \text{ in } Q \\ \\ z_{tt} - \partial_i(b_{ij}(x)\partial_j z) + a(x,t)z = 0 \text{ in } Q \\ p = 0, \quad z = \frac{\partial p}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma \\ p(T) = p^0; \quad p_t(T) = p^1, \quad z(T) = z^0; \quad z_t(T) = z^1 \text{ in } \Omega \end{cases}$$

For $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$, we have $p \in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega))$, and $z \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; H^{-1}(\Omega))$. For every $t \in [0,T]$, define the energy

$$E(p;t) = \frac{1}{2}\left(|p_t(.,t)|^2_{L^2(\Omega)} + \int_\Omega (b_{ij}(x)\partial_j p(x,t)\partial_i p(x,t)\,dx\right).$$

Thanks to Lions' H.U.M, the linear controllability problem will be solved once we prove the following observability estimate:

Proposition

Let ω be a neighborhood of Γ_0, and let $T > 2R_1$. Let $\varepsilon > 0$ with $(N-2)\varepsilon < 4$. There exists

$$K_\varepsilon = \exp\left[C_\varepsilon(1 + ||a||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta_\varepsilon}} + ||b||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta_\varepsilon}})\right]$$

such that for all $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and all $(z^0, z^1) \in L^2(\Omega) \times H^{-1}(\Omega)$:

$$E(p; T) + \widehat{E}(z; T) \leq K_\varepsilon \int_0^T \int_\omega |z(x,t)|^2 \, dxdt,$$

where $C_\varepsilon = C_\varepsilon(\varepsilon, \Omega, \omega, T, b_{ij}) > 0$, $l_\varepsilon = 2 + 4\varepsilon^{-1}$, $\theta_\varepsilon = \varepsilon N/(4 + 2\varepsilon)$, and $||.||_{\infty,r} = ||.||_{L^\infty(0,T;L^r(\Omega))}$.

Proof of Proposition: key elements

Step 1. Establish the energy estimate

$$E(p;t) \leq E(p;s)\exp\left(C_\varepsilon\left(1+||b||_{\infty,l_\varepsilon}^{\frac{1+\theta_\varepsilon}{2}}\right)|t-s|\right), \quad \forall s,t \in [0,T].$$

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Step 2. Use the Duyckaerts-Zhang-Zuazua (boundary) Carleman estimate and Step 1 to derive the boundary observability estimate

$$E(p;T) \leq e^{C_\varepsilon(1+||b||_{\infty,I_\varepsilon}^{\frac{2}{3-2\theta_\varepsilon}})} \int_0^T r^2 \int_{\Gamma_0} \left|\frac{\partial p(\gamma,t)}{\partial \nu_B}\right|^2 d\gamma dt.$$

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Step 3. Use a localizing argument to derive the partial estimate

$$E(p;T) \leq K_\varepsilon \int_0^T \int_\omega |z(x,t)|^2 \, dxdt.$$

Step 4. Use the Duyckaerts-Zhang-Zuazua internal observability estimate to get

$$\widehat{E}(z;T) \leq K_\varepsilon \int_0^T \int_\omega |z(x,t)|^2\, dxdt + C(\Omega, T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial p(\gamma, t)}{\partial \nu_B} \right|^2 d\gamma dt.$$

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Step 5. Use Lions'inequality

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial p(\gamma,t)}{\partial \nu_B}\right|^2 d\gamma dt \leq K_\varepsilon E(p;T),$$

in Step 4, and combine the result with Step 3 to get the claimed estimate. □

1. Do we have $E(u;0) + \widehat{E}(v;0) \leq C \int_0^T \int_\omega |u_t(x,t)|^2 \, dxdt$, with no smallness assumption on the couplings?

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3. What about different principal operators? An improved version of Theorem 2 is known to hold for the heat equation, but its boundary counterpart fails in general (wave and heat). A similar result holds for two wave equations coupled in cascade internally when the two operators are proportional & Ω is a compact C^∞ manifold with no boundary; in particular it is shown by Dehman-Leautaud-Lerousseau that if $\omega \cap \mathcal{O}$ satisfies GCC, then:

$$\widehat{E}(u;0) + E_{-2}(v;0) \leq C \int_0^T \int_\omega |u(x,t)|^2 \, dxdt,$$

where $2E_{-2}(v;0) = ||v^0||^2_{H^{-2}(\Omega)} + ||v^0||^2_{H^{-3}(\Omega)}$.

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4. What about other boundary conditions?

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!