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CONTROLLABILITY OF COUPLED SYSTEMS
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Desensitizing controls for some hyperbolic equations

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Notations.

$\Omega =$ bounded domain in $\mathbb{R}^d$, $d \geq 1$,
$\Gamma =$ boundary of (Ω) class C^2, $\nu =$unit normal pointing into the exterior of Ω.
Γ_0=nonvoid open set in Γ.
$T > 0$, $Q = \Omega \times (0, T)$, $\Sigma_0 = \Gamma_0 \times (0, T)$,
$\mathcal{O}$, $\omega =$ nonvoid open sets in Ω.
Einstein summation convention used.

Motivation. Consider the controllability problem: Given $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $\xi \in L^2(Q)$, find a control $v \in L^2(0,T; L^2(\omega))$ such that the solution y of the hyperbolic system:

$$\begin{cases} y_{tt} - \partial_i(b_{ij}(x)\partial_j y) = \xi + v1_\omega \text{ in } Q \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y(0) = y^0, \quad y_t(0) = y^1 \text{ in } \Omega, \end{cases}$$

where the coefficients $(b_{ij})_{i,j}$, satisfy:

$$b_{ij} \in C^1(\bar{\Omega}); \quad b_{ij} = b_{ji}, \quad \forall i, j = 1, 2, ..., d,$$

and $\exists b_0 > 0$:

$$b_{ij}(x) z_i z_j \geq b_0 z_i z_i, \quad \forall (x, z) \in \bar{\Omega} \times \mathbb{R}^d,$$

then y satisfies:

$$y(x, T) = 0, \quad y_t(x, T) = 0 \text{ in } \Omega.$$

Solving this controllability problem amounts to showing that for the adjoint system:

$$
\begin{aligned}
&u_{tt} - \partial_i(b_{ij}(x)\partial_j u) = 0 \text{ in } Q \\
&u = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\
&u(0) = u^0 \in L^2(\Omega), \quad u_t(0) = u^1 \in H^{-1}(\Omega),
\end{aligned}
$$

one has the observability estimate:

$$
||u^0||^2_{L^2(\Omega)} + ||u^1||^2_{H^{-1}(\Omega)} \leq C \int_0^T \int_\omega |u|^2 \, dx dt.
$$

This controllability problem may be solved for large enough time T, and control set ω with the help of the Hilbert uniqueness method (HUM) of Lions (e.g. Haraux (1988), Bardos-Lebeau-Rauch, Zuazua in Lions' Book on controllability (1988), Fursikov-Imanuvilov (1996), Yao (1999), Zhang (2000)). In particular, ω must satisfy the Bardos-Lebeau-Rauch geometric control condition.

Such a control v is very sensitive to small variations of the data of the system; so we cannot expect a quantity such as $\int_0^T \int_{\Gamma_0} |\frac{\partial y(\gamma,t)}{\partial \nu_B}|^2 \, d\gamma dt$ to be insensitive to such variations. For the latter to happen, a new concept of controllability is needed. This is probably what motivated Lions to introduce the notion of desensitizing control in the late eighties.

Desensitizing control problem: Linear case. Consider the new control problem: Given $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, and $\xi \in L^2(Q)$, find a control $v \in L^2(0, T; L^2(\omega))$ such that for the solution y of the hyperbolic system:

$$
\begin{cases}
y_{tt} - \partial_i(b_{ij}(x)\partial_j y) = \xi + v1_\omega \text{ in } Q \\
y = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\
y(0) = y^0 + \tau_0 \hat{y}^0 \\
y_t(0) = y^1 + \tau_1 \hat{y}^1 \text{ in } \Omega,
\end{cases}
$$

- $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ have unit norm, are arbitrary and unknown,
- τ_0 and τ_1 are arbitrary small unknown real numbers,

the quantity

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial y(\gamma, t)}{\partial \nu_B} \right|^2 d\gamma dt$$

is insensitive to small perturbations of the initial data. In other words, can we find a control v that desensitizes the functional Φ defined by

$$\Phi(y) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y(\gamma, t)}{\partial \nu_B} \right|^2 d\gamma dt?$$

This amounts to constructing a control v such that for all $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ with unit norm:

$$\left. \frac{\partial \Phi(y)}{\partial \tau_0} \right|_{\tau_0=\tau_1=0} = 0 = \left. \frac{\partial \Phi(y)}{\partial \tau_1} \right|_{\tau_0=\tau_1=0}.$$

History: desensitizing control

a) J.L. Lions (1989), parabolic equations.
b) Bodart-Fabre (ε-desensitizing controls, 1995),
c) de Teresa (1997, 2000),
d) Bodart-Gonzalez Burgos-Perez Garcia (3 papers, 2004),
e) Fernandez Cara-Garcia-Osses (2005),
f) de Teresa-Zuazua (2009),
g) Kavian-de Teresa (2010)...

h) Desensitizing control concept in its infancy for second order evolution equations,
i) Dáger (2006), desensitizing controls, one dimensional wave equation. The proof technique developed by Dáger critically relies on the fact that the one dimensional wave equation is time periodic, which is not the case in higher dimensions.

The functional that the control desensitizes in Dáger paper, as well as in almost all the papers dealing with parabolic equations is

$$\Psi(y) = \frac{1}{2} \int_0^T \int_{\mathcal{O}} |y(x,t)|^2 \, dxdt,$$

where $\mathcal{O}$ is another open subset of Ω.

Theorem 1. *A control v desensitizes the functional Φ if and only if the solution pair (y_0, q) of the cascade system:*

$$\begin{cases} y_{0tt} - \partial_i(b_{ij}(x)\partial_j y_0) = \xi + v\chi_\omega \ \textit{in } Q \\ y_0 = 0 \ \textit{on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \ \textit{in } \Omega \end{cases}$$

$$\begin{cases} q_{tt} - \partial_i(b_{ij}(x)\partial_j q) = 0 \ \textit{in } Q \\ q = \dfrac{\partial y_0}{\partial \nu_B}\chi_{\Gamma_0} \ \textit{on } \Sigma = \partial\Omega \times (0,T) \\ q(T) = 0; \quad q_t(T) = 0 \ \textit{in } \Omega, \end{cases}$$

satisfies:

$$q(0) = 0, \qquad q_t(0) = 0.$$

Theorem. 2. *Suppose that the coefficients* b_{ij} *are* C^2*, and* Ω *is* C^3*. Let* Γ_0 *satisfy the geometric control condition of Bardos-Lebeau-Rauch. Assume that* ω *is a neighborhood of* Γ_0*. There exists a positive time* T_0 *depending only on* Ω *and* Γ_0 *such that for every* $T > T_0$*, and for all* $y^0 \in H_0^1(\Omega)$ *and* $y^1 \in L^2(\Omega)$*, there exists a control* $v \in L^2(0, T; L^2(\omega))$ *that desensitizes the functional* Φ.

Thanks to Theorem 1 and Lions' HUM, proving Theorem 2 amounts to proving the following observability inequality:

$$||p^0||^2_{H^1_0(\Omega)} + ||p^1||^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |z|^2 \, dxdt,$$

for the adjoint cascade system:

$$\begin{cases} p_{tt} - \partial_i(b_{ij}(x)\partial_j p) = 0 \text{ in } Q \\ p = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ p(0) = p^0; \quad p_t(0) = p^1 \text{ in } \Omega \end{cases}$$

$$\begin{cases} z_{tt} - \partial_i(b_{ij}(x)\partial_j z) = 0 \text{ in } Q \\ z = \dfrac{\partial p}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ z(T) = 0; \quad z_t(T) = 0 \text{ in } \Omega. \end{cases}$$

To prove the observability inequality:

$$||p^0||^2_{H^1_0(\Omega)} + ||p^1||^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |z|^2 \, dxdt,$$

first we show the weighted observability inequality [Burq, 1997]:

$$E(p;0) \leq C \int_0^T r(t)^2 \int_{\Gamma_0} |\frac{\partial p(\gamma,t)}{\partial \nu_B}|^2 \, d\gamma dt.$$

Then, we use a localizing argument.

ε-desensitizing controls. If the observation set Γ_0 does not satisfy the GCC, then we cannot expect to build desensitizing controls, but we can construct ε-desensitizing controls. A control v is said to ε-desensitize the functional Φ defined by:

$$\Phi(y) = \frac{1}{2} \int_0^T \int_{\Gamma_0} \left| \frac{\partial y(\gamma, t)}{\partial \nu_B} \right|^2 d\gamma dt$$

if for all $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ with unit norm:

$$\forall \varepsilon > 0, \quad \left| \frac{\partial \Phi(y)}{\partial \tau_0} \Big|_{\tau_0 = \tau_1 = 0} \right| \leq \varepsilon,$$

and

$$\left| \frac{\partial \Phi(y)}{\partial \tau_1} \Big|_{\tau_0 = \tau_1 = 0} \right| \leq \varepsilon.$$

Theorem 3. *A control v ε-desensitizes the functional Φ if and only if the solution pair (y_0, q) of the cascade system:*

$$\begin{cases} y_{0tt} - \partial_i(b_{ij}(x)\partial_j y_0) = \xi + v\chi_\omega \ \textit{in } Q \\ y_0 = 0 \ \textit{on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \ \textit{in } \Omega \end{cases}$$

$$\begin{cases} q_{tt} - \partial_i(b_{ij}(x)\partial_j q) = 0 \ \textit{in } Q \\ q = \dfrac{\partial y_0}{\partial \nu_B}\chi_{\Gamma_0} \ \textit{on } \Sigma = \partial\Omega \times (0,T) \\ q(T) = 0; \quad q_t(T) = 0 \ \textit{in } \Omega, \end{cases}$$

satisfies:

$$||q(0)||_{L^2(\Omega)} \leq \varepsilon, \quad ||q_t(0)||_{H^{-1}(\Omega)} \leq \varepsilon.$$

Theorem. 4. *Suppose that the coefficients* b_{ij} *are* C^2, *and* Ω *is* C^3. *Let* Γ_0 *be a non-void open subset of the boundary of* Ω. *Assume that* ω *is a neighborhood of* Γ_0. *There exists a positive time* T_0 *depending only on* Ω *and* Γ_0 *such that for every* $T > T_0$, *and for all* $y^0 \in H_0^1(\Omega)$ *and* $y^1 \in L^2(\Omega)$, *there exists a control* $v \in L^2(0,T;L^2(\omega))$ *that* ε-*desensitizes the functional* Φ.

It may be shown that proving Theorem 4 reduces to proving the following observability inequality:

$$
\begin{aligned}
&||p^0||^2_{L^2(\Omega)} + ||p^1||^2_{H^{-1}(\Omega)} \\
&\leq C \left(\ln \left(2 + \frac{E(p;0)}{\int_0^T \int_\omega |z|^2 \, dxdt} \right) \right)^{-1} E(p;0)
\end{aligned}
$$

for the adjoint cascade system:

$$\begin{cases} p_{tt} - \partial_i(b_{ij}(x)\partial_j p) = 0 \text{ in } Q \\ p = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ p(0) = p^0; \quad p_t(0) = p^1 \text{ in } \Omega \end{cases}$$

$$\begin{cases} z_{tt} - \partial_i(b_{ij}(x)\partial_j z) = 0 \text{ in } Q \\ z = \dfrac{\partial p}{\partial \nu_B}\chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ z(T) = 0; \quad z_t(T) = 0 \text{ in } \Omega. \end{cases}$$

To prove the observability inequality:

$$||p^0||^2_{L^2(\Omega)} + ||p^1||^2_{H^{-1}(\Omega)}$$
$$\leq C \left(\ln \left(2 + \frac{E(p;0)}{\int_0^T \int_\omega |z|^2 \, dx dt} \right) \right)^{-1} E(p;0)$$

first we derive the weak observability inequality [Robbiano, 1995]:

$$||p^0||^2_{L^2(\Omega)} + ||p^1||^2_{H^{-1}(\Omega)}$$
$$\leq C\frac{E(p;0)}{\lambda}$$
$$+ Ce^{\mu\lambda}\int_0^T r(t)^2 \int_{\Gamma_0} |\frac{\partial p(\gamma,t)}{\partial \nu_B}|^2 \, d\gamma dt.$$

Then, we use a localizing argument.

Internal observation. Let $\mathcal{O}$ be a nonvoid open subset of Ω. A control v desensitizes (resp. ε-desensitizes) the functional Ψ defined by

$$\Psi(y) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |y(x,t)|^2 \, dxdt,$$

if for all $\hat{y}^0$ and $\hat{y}^1$ with unit norm in appropriate Hilbert spaces, one has:

$$\left.\frac{\partial \Psi(y)}{\partial \tau_0}\right|_{\tau_0=\tau_1=0} = 0 = \left.\frac{\partial \Psi(y)}{\partial \tau_1}\right|_{\tau_0=\tau_1=0},$$

respectively:

$$\forall \varepsilon > 0, \quad \left| \left.\frac{\partial \Psi(y)}{\partial \tau_0}\right|_{\tau_0=\tau_1=0} \right| \leq \varepsilon,$$

$$\left| \left.\frac{\partial \Psi(y)}{\partial \tau_1}\right|_{\tau_0=\tau_1=0} \right| \leq \varepsilon.$$

Theorem 5. *A control v desensitizes the functional Ψ if and only if the solution pair (y_0, q) of the cascade system:*

$$\begin{cases} y_{0tt} - \partial_i(b_{ij}(x)\partial_j y_0) = \xi + v1_\omega \ \ in \ Q \\ y_0 = 0 \ \ on \ \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \ \ in \ \Omega \end{cases}$$

$$\begin{cases} q_{tt} - \partial_i(b_{ij}(x)\partial_j q) = y1_{\mathcal{O}} \ \ in \ Q \\ q = 0 \ \ on \ \Sigma = \partial\Omega \times (0,T) \\ q(T) = 0; \quad q_t(T) = 0 \ \ in \ \Omega, \end{cases}$$

satisfies:

$$q(0) = 0, \qquad q_t(0) = 0.$$

Theorem. 6. *Suppose that the coefficients* b_{ij} *are* C^3*, and* Ω *is also* C^3*. Let* $\omega_0 \subset\subset \mathcal{O} \cap \omega$ *be an open subset satisfying the geometric control condition of Bardos-Lebeau-Rauch. There exists a positive time* T_0 *depending only on* Ω *and* ω_0 *such that for every* $T > T_0$*, and for all* $y^0 \in L^2(\Omega)$ *and* $y^1 \in H^{-1}(\Omega)$*, there exists a control* $v \in [H^1(0,T;L^2(\omega))]'$ *that desensitizes the functional* Ψ*.*

Thanks to Theorem 5 and Lions' HUM, proving Theorem 2 amounts to proving the following observability inequality:

$$||p^0||^2_{L^2(\Omega)}+||p^1||^2_{H^{-1}(\Omega)} \leq C\int_0^T\int_\omega |z_t|^2\, dxdt,$$

for the adjoint cascade system:

$$\begin{cases} p_{tt} - \partial_i(b_{ij}(x)\partial_j p) = 0 \text{ in } Q \\ p = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ p(0) = p^0; \quad p_t(0) = p^1 \text{ in } \Omega \end{cases}$$

$$\begin{cases} z_{tt} - \partial_i(b_{ij}(x)\partial_j z) = p1_{\mathcal{O}} \text{ in } Q \\ z = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ z(T) = 0; \quad z_t(T) = 0 \text{ in } \Omega. \end{cases}$$

To prove the observability inequality:

$$||p^0||^2_{H^1_0(\Omega)} + ||p^1||^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega |z_t|^2 \, dxdt,$$

first we show the weighted observability inequality:

$$E(p;0) \leq C \int_0^T r(t)^2 \int_{\omega_0} |p(x,t)|^2 \, dxdt.$$

Then, we use a localizing argument.

Theorem 7. *Suppose that* $\mathcal{O}$ *and* ω *are two nonempty open sets in* Ω *with* $\mathcal{O} \cap \omega \neq \emptyset$*. There exists a positive time* T^* *depending only on* Ω*,* $\mathcal{O}$*, and* ω *such that for every* $T > T^*$*, for all* $y^0 \in H_0^1(\Omega)$ *and* $y^1 \in L^2(\Omega)$*, and for every positive constant* ε*, there exists a control* $v \in L^2(0, T; L^2(\omega))$ *that* ε*-desensitizes the functional* Ψ*.*

To prove Theorem 7, it is enough to show the following observability inequality:

$$||p^0||^2_{L^2(\Omega)} + ||p^1||^2_{H^{-1}(\Omega)} \leq C \left(\ln \left(2 + \frac{1}{\int_0^T \int_\omega |z|^2 \, dxdt} \right) \right)^{-1} E(p;0)$$

To prove the observability inequality, first we derive the weak observability inequality [Bellassoued, 2005]:

$$
\begin{aligned}
&||p^0||^2_{L^2(\Omega)} + ||p^1||^2_{H^{-1}(\Omega)} \\
&\leq C\frac{E(p;0)}{\lambda} \\
&+ Ce^{\mu\lambda}\int_0^T r(t)^2 \int_{\omega_0} |p(x,t)|^2 \, dxdt.
\end{aligned}
$$

Then, we use a localizing argument.

A nonlinear problem. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuously differentiable function with

$$\limsup_{|s|\to\infty} \frac{|f(s)|}{|s|(\log|s|)^{\alpha}} = 0,$$

for some $0 \leq \alpha < 3/2$.

Let $\xi \in L^2(Q)$, and consider the wave equation:

$$\begin{cases} y_{tt} - \Delta y + f(y) = \xi + v\chi_\omega \text{ in } Q \\ y = 0 \text{ on } \Sigma = \partial\Omega \times (0, T) \\ y(0) = y^0 + \tau_0 \hat{y}^0 \\ y_t(0) = y^1 + \tau_1 \hat{y}^1 \text{ in } \Omega \end{cases}$$

where $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ are given, $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ have unit norm and are unknown, and τ_0 and τ_1 are small unknown real numbers.

We want to find a control v that desensitizes the functional Φ defined by

$$\Phi(y) = \frac{1}{2}\int_0^T \int_{\Gamma_0} |\frac{\partial y(\gamma, t)}{\partial \nu}|^2 \, d\gamma dt.$$

This amounts to constructing a control v that satisfies for all $(\hat{y}^0, \hat{y}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ with unit norm:

$$\left.\frac{\partial \Phi(y)}{\partial \tau_0}\right|_{\tau_0=\tau_1=0} = 0 = \left.\frac{\partial \Phi(y)}{\partial \tau_1}\right|_{\tau_0=\tau_1=0}.$$

History: controllability of semilinear wave equation

a) Fattorini (1975), 1d hyperbolic equations, implicit function method,

b) Chewning (1976) generalizes Fattorini (1975) to higher dimensions,

c) Zuazua (1990-1991), HUM + Schauder fixed-point (linear growth)

d) Zuazua (1993), HUM + Leray-

Schauder (1d, superlinear growth allowed)

e) Lasiecka-Triggiani (1991), global inversion theorem (Lipschitz),

f) Cannarsa-Komornik-Loreti (1999), 1d, iterated log, improves Zuazua (1993)

g) Li-Zhang (2000), Carleman estimates, superlinear growth allowed,

h) Martinez-Vancostenoble (2003), 1d, arbitrarily short time,

i) Fu-Yong-Zhang (2007), Carleman estimates, hyperbolic equations,

j) Duyckaerts-Zhang-Zuazua (2008), improved Carleman estimates, allows

$$\limsup_{|s|\to\infty} \frac{|f(s)|}{|s|(\log|s|)^\alpha} = 0, \quad 0 \le \alpha < 3/2.$$

Theorem. 8. *Let* $x_0 \in \mathbb{R}^d \setminus \bar{\Omega}$. *Set* $\Gamma_0 = \{x \in \Gamma; (x - x_0) \cdot \nu > 0\}$. *Assume that* ω *is a neighborhood of* Γ_0. *There exists a positive time* T_0^* *depending only on* Ω *and* ω *such that for every* $T > T_0^*$, *and for all* $y^0 \in H_0^1(\Omega)$ *and* $y^1 \in L^2(\Omega)$, *there exists a control* $v \in L^2(0, T; L^2(\omega))$ *that desensitizes the functional* Φ.

The main ingredient for proving the existence of a desensitizing control is to reduce the problem to a controllability problem. To this end, consider the following cascade wave equations:

$$\begin{cases} y_{0tt} - \Delta y_0 + f(y_0) = \xi + v\chi_\omega \text{ in } Q \\ y_0 = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \text{ in } \Omega \end{cases}$$

$$\begin{cases} q_{tt} - \Delta q + f'(y_0)q = \text{ in } Q \\ q = \dfrac{\partial y_0}{\partial \nu}\chi_{\Gamma_0} \text{ on } \Sigma = \partial\Omega \times (0,T) \\ q(T) = 0; \quad q_t(T) = 0 \text{ in } \Omega. \end{cases}$$

Proposition 9. *A control v desensitizes the functional* Ψ if and only if the solution pair (y_0, q) satisfies:

$$q(0) = 0, \qquad q_t(0) = 0.$$

Set

$$g(s) = \begin{cases} (f(s) - f(0))/s, \text{ if } s \neq 0 \\ f'(0), \text{ if } s = 0. \end{cases}$$

Let $w \in L^\infty(0, T; L^2(\Omega))$. Set

$$a(x,t) = g(w(x,t)), \quad b(x,t) = f'(w(x,t)).$$

The nonlinear cascade system may be linearized as:

$$\begin{cases} y_{0tt} - \Delta y_0 + a y_0 = -f(0) + \xi + v\chi_\omega \text{ in } Q \\ y_0 = 0 \text{ on } \Sigma \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \text{ in } \Omega \end{cases}$$

$$\begin{cases} q_{tt} - \Delta q + b q = 0 \text{ in } Q \\ q = \dfrac{\partial y_0}{\partial \nu}\chi_{\Gamma_0} \text{ on } \Sigma \\ q(T) = 0; \quad q_t(T) = 0 \text{ in } \Omega, \end{cases}$$

Introduce the adjoint system:

$$\begin{cases} p_{tt} - \Delta p + bp = 0 \text{ in } Q \\ p = 0 \text{ on } \Sigma \\ p(0) = p^0; \quad p_t(0) = p^1 \text{ in } \Omega \end{cases}$$

$$\begin{cases} z_{tt} - \Delta z + az = 0 \text{ in } Q \\ z = \dfrac{\partial p}{\partial \nu} \chi_{\Gamma_0} \text{ on } \Sigma \\ z(T) = 0; \quad z_t(T) = 0 \text{ in } \Omega. \end{cases}$$

Sketch of the proof of Theorem 8. Thanks to Lions' HUM, the proof of Theorem reduces to proving:

Proposition 10. *Let* ω*, and* T *be given as in Theorem. Let* $\varepsilon > 0$ *with* $(d-2)\varepsilon < 4$*. There exists*

$$C_1 = \exp C_0(1 + ||a||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta}} + ||b||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta}})$$

such that for all $(p^0, p^1) \in H_0^1(\Omega) \times L^2(\Omega)$*:*

$$E(p;0) \leq C_1 \int_0^T \int_\omega |z(x,t)|^2 \, dxdt,$$

where $l_\varepsilon = 2 + 4\varepsilon^{-1}$, and $\theta = \varepsilon d/(4 + 2\varepsilon)$, and $||.||_{\infty,r} = ||.||_{L^\infty(0,T;L^r(\Omega))}$.

Thanks to Proposition 10 and Lions' HUM, the linearized problem is exactly controllable. A Schauder fixed-point argument shows that the nonlinear system is also exactly controllable; the exact control v desensitizes the functional Ψ by Proposition 9. $\square$

Sketch of the proof of Proposition 10.

• Apply the [Duyckaerts-Zhang-Zuazua] Carleman estimate to get: the existence of positive constants C, μ, and $\lambda_\varepsilon = C(1 + ||b||_{\infty,l_\varepsilon}^{\frac{2}{3-2\theta}})$ such that for all $\lambda \geq \lambda_\varepsilon$:

$$\int_Q r^2\{\lambda^3|p(x,t)|^2 + \lambda|p_t|^2 + \lambda|\nabla p|^2\}\,dxdt$$

$$\leq C\lambda e^{-\mu\lambda}E(p;0)$$

$$+ Ce^{C\lambda}\int_0^T r^2\int_{\Gamma_0}|\frac{\partial p(\gamma,t)}{\partial\nu}|^2\,d\gamma dt.$$

• Show that for all $s,\ t \in [0,T]$, one has:

$$E(p;t) \leq E(p;s)\exp(C(1 + ||b||_{\infty,l_\varepsilon}^{\frac{1+\theta}{2}})|t-s|).$$

- Derive the inverse inequality:

$$E(p;0) \le [\exp \lambda_\varepsilon] \int_0^T r^2 \int_{\Gamma_0} |\frac{\partial p(\gamma,t)}{\partial \nu}|^2 \, d\gamma dt.$$

- Use a localizing argument to get:

$$\int_0^T r^2 \int_{\Gamma_0} |\frac{\partial p(\gamma,t)}{\partial \nu}|^2 \, d\gamma dt$$
$$\le C(1+||b-a||^2_{\infty,l_\varepsilon}) \int_0^T \int_\omega |z|^2 \, dxdt.$$

- Combine both estimates to find:

$$E(p;0) \le C(1+||a||^2_{\infty,l_\varepsilon})[\exp \lambda_\varepsilon] *$$
$$\int_0^T \int_\omega |z|^2 \, dxdt.$$

Final remarks and open problems. 1) The method developed to solve the desensitizing control problems may be used to construct a desensitizing control v such that the solution of the system:

$$\begin{cases} y_{0tt} - \Delta y_0 + f(y_0) = \xi + v\chi_\omega \text{ in } Q \\ y_0 = 0 \text{ on } \Sigma = \partial\Omega \times (0,T) \\ y_0(0) = y^0; \quad y_{0t}(0) = y^1 \text{ in } \Omega, \end{cases}$$

is steered to some prescribed final state.

2) In the multidimensional setting, can we build ε-desensitizing controls for hyperbolic equations when $\mathcal{O} \cap \omega = \emptyset$?

1d case solved (ordinary wave equation), (Dager, 2006). Parabolic equations, solved (Kavian-de Teresa, 2010).

3) For which class of initial data can controls $v \in L^2(0,T;L^2(\omega))$ be built so as to desensitize the functional

$$\Psi(y) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |y(x,t)|^2 \, dxdt,$$

for hyperbolic equations?

Problem solved for parabolic equations by de Teresa-Zuazua (2009).

4) Can boundary desensitizing controls be built in the multidimensional setting?

boundary ε-desensitizing controls for parabolic equations found in Bodart-Fabre (1995) when the control and observation sets are intersecting boundary portions. See also Kavian-de Teresa (2010) for more general results in the same framework.

5) Constructing desensitizing controls for plate equations or coupled systems (thermoelasticity, maxwell equations,...) remains a widely open problem.