Florida International University

MAC 2312 (Calculus II)
Integration Problems

Be sure to be able to do all these problems by the end of the semester.

A. Compute the following integrals.

1) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4 x\,dx$, 2) $\int_0^{\frac{\pi}{2}} \cos^3 x \sin^2 x\,dx$, 3) $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2}+\cos x}\,dx$, 4) $\int_{-1}^{1}(\arcsin x)^2\,dx$, 5) $\int_0^1 \frac{1-\sqrt{x}}{1-\sqrt[6]{x}}\,dx$

6) $\int_0^{\frac{\pi}{6}}(\sin^6 t+\cos^6 t)\,dt$, 7) $\int_0^{\frac{\pi}{4}} \frac{\cos(2x)dx}{2-\sin^2(2x)}$, 8) $\int_0^{2\pi} \frac{\sin^2 u du}{\cos^4 u+\sin^4 u}$, 9) $\int_0^{\pi} \frac{dx}{3+\cos x}$, 10) $\int_0^1 \arctan(\sqrt{1-x^2})\,dx$.

11) Let $a<b$ be two real numbers. Let $f:[a,b]\mapsto\mathbb{R}$ be a continuous function with $f(a+b-x)=f(x)$ for every x in $[a,b]$. a) Show that $\int_a^b xf(x)\,dx = \frac{a+b}{2}\int_a^b f(x)\,dx$. b) Use part a) to evaluate $I=\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$.

12) Set $I=\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin x\cos x}}\,dx$ and $J=\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sqrt{1+\cos x\sin x}}\,dx$. a) Show that $I=J$. b) Use appropriate trigonometric identities to evaluate $I+J$, then derive the values of I and J.

13) $\int_0^{\frac{\pi}{2}} \frac{x\sin x\cos x}{\tan^2 x+\cot^2 x}\,dx$, 14) $\int_0^{\frac{\pi}{3}} \frac{1}{1+\cos t}\,dt$, 15) $\int_0^{\pi} \frac{x}{2+\sin x}\,dx$, 16) $\int_0^1 x\sqrt{\frac{1-x}{x+1}}\,dx$, 17) $\int_{\frac{1}{2}}^{1} \frac{dx}{(1+x)\sqrt{1-x^2}}$,

18) $\int_0^4 \sqrt{x(4-x)}\,dx$, 19) $\int_0^1 \frac{\sqrt{3-x^2}}{4+x}\,dx$, 20) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x}+\sqrt{1+x}}$, 21) $\int \frac{dx}{1+\sin^2 x}$, 22) $\int \frac{x}{\sqrt{12-12x-9x^2}}\,dx$

23) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(1+\cos x)(1+\cos x+\sin x)}\,dx$, 24) $\int_0^{\frac{\pi}{4}} \frac{\tan x}{1+\cos x}\,dx$, 25) $\int_0^1 \frac{\ln x}{x+1}\,dx$, 26) $\int_{-1}^{1} \frac{3x^3}{1+5^x}\,dx$.

B. Area between curves.

1) Find the area of the region bounded by the curves $y=\frac{\sqrt{x}}{x+1}$, $x=0$, and $x=1$.

2) One wants to find the area under the curve $y=\sin^2 x$ for $0\le x\le \pi/2$. a) Use the trigonometric complementary identity to show that $\int_0^{\frac{\pi}{2}}\sin^2 x\,dx=\int_0^{\frac{\pi}{2}}\cos^2 x\,dx$. (Do not evaluate any of those integrals). b) Use the trigonometric identity $\sin^2 x+\cos^2 x=1$ to show that the value of that area is $\pi/4$.

3) Find the area of the region bounded by the curves $y=\frac{1}{1+e^x}$, $y=e^{-2x}$, and the y-axis.

4) Find the area of the region in the first quadrant enclosed by the curves $y=\sqrt[3]{x}$ and $y=\frac{6}{1+\sqrt[3]{x}}$.

5) Find the area of the region in the first quadrant enclosed by the curves $y=\frac{1}{1+\sin x}$, $y=\frac{1}{1+\cos x}$, and the y-axis.

C. Riemann sums.

1) Let f be continuous on (a,b) such that the improper integral $\int_a^b f(x)\,dx$ converges. a) Show that the sequence $(S_n)_{n\ge 2}$ given by $S_n=\frac{b-a}{n}\sum_{k=1}^{n-1} f(a+k\frac{b-a}{n})$ converges to $\int_a^b f(x)\,dx$. b) Given that for each $n\ge 2$, one has: $\sin(\frac{\pi}{n})\sin(\frac{2\pi}{n})...\sin(\frac{(n-1)\pi}{n})=\frac{n}{2^{n-1}}$, use Riemann sums to evaluate the integral $\int_0^{\pi}\ln(\sin u)\,du$.

2) Use Riemann sums to evaluate each of the following limits

a) $\lim_{n\to\infty}\frac{1}{n\sqrt{n}}\sum_{k=1}^{n}\sqrt{k}$, b) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n+k}$, c) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{n^2+k^2}$, d) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\sin(\frac{k\pi}{n})$, e) $\lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$,

f) $\lim_{n\to\infty}\left[\prod_{k=1}^{n}(1+\frac{k}{n})^k\right]^{\frac{1}{n^2}}$, g) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}\sin(\frac{k\pi}{n+1})$, h) $\lim_{n\to\infty}\left(\left[\sum_{k=1}^{n}e^{\frac{1}{n+k}}\right]-n\right)$ i) $\lim_{n\to\infty}\sum_{k=n}^{2n}\frac{1}{k}$,

j) $\lim_{n\to\infty}\left(1-n\sum_{k=1}^{n}\frac{1}{n^2+k}\right)n$, k) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{\sqrt{k^2+n^2+n+k}}$, l) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}f(a+\frac{k(b-a)}{n^2})$ where f is continuously differentiable on $[a,b]$. Discuss the case where f is continuous only on $[a,b]$,

m) $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n}f(a+\frac{k(b-a)}{n^2})$ where f is twice continuously differentiable on $[a,b]$ with $f(a)=0$,

n) $\lim_{n\to\infty}\sum_{k=1}^{n}\sin(\frac{k\pi}{n})\sin(\frac{k\pi}{n^2})$.

D. Indefinite integrals. Evaluate each of the following integrals.

1) $\int \frac{1+\sin x}{1-\cos x}\,dx$, 2) $\int \frac{\sin^3 x}{(1+\cos x)^2}\,dx$, 3) $\int \frac{\cos(3x)}{\cos(2x)-5}\,dx$, 4) $\int \frac{\sin x-\cos x}{1+\cos x}\,dx$, 5) $\int \frac{2\csc x+3\cot x}{3+4\cot x}\,dx$, 6) $\int \frac{1+e^x}{1+e^{2x}}\,dx$

7) $\int x(\arctan x)^2\,dx$, 8) $\int \frac{e^{2x}+4e^x+1}{e^{3x}-1}\,dx$, 9) $\int \frac{x^{\frac{2}{3}}+x^{\frac{5}{6}}}{1+\sqrt{x}}\,dx$, 10) $\int \frac{\sqrt{x}+\sqrt[3]{x}}{1+\sqrt[4]{x}}\,dx$ 11) $\int \frac{1}{x}\sqrt{\frac{x-1}{x}}\,dx$, 12) $\int \frac{x^4}{(1-x^2)^{\frac{3}{2}}}\,dx$,

13) $\int \arctan(\sqrt[3]{x})\,dx$, 14) $\int \frac{dx}{\sqrt{1+e^x}}$, 15) $\int \frac{(t+3)dt}{(t^2+t+1)^2}$, 16) $\int \frac{x^2\ln x}{(x^3+1)^3}\,dx$, 17) $\int \frac{dx}{\sin x-\cos x+\sqrt{2}}$, 18) $\int \frac{dx}{\cos^2 x\sin^4 x}$

19) $\int \frac{xe^x}{\sqrt{1+e^x}}\,dx$, 20) $\int \frac{\sin x}{\sin^3 x+\cos^3 x}\,dx$, 21) $\int \frac{x^4}{(1-x^2)^{\frac{5}{2}}}\,dx$.

E. Volumes.

1) Find the volume of the solid obtained by revolving around the $x-$axis the region bounded by the curve $y=\frac{x}{x^2+1}$, $0\le x\le 1$.

2) Find the volume of the solid obtained by revolving the region enclosed by the curve in 1), and the curves $x=0$, and $y=1/2$ around the $y-$axis.

3) Find the volume of the solid obtained by revolving around the $x-$axis the region bounded by the curve $y=\frac{1}{1+e^x}$, $0\le x\le 1$.

4) Find the volume of the solid obtained by revolving around the $y-$axis the region bounded by the curve $y=\frac{1}{2+\sin x}$, $0\le x\le \pi$.

5) Find the volume of the solid obtained by revolving around the $x-$axis the region bounded by the curve $y=\sqrt{\frac{1+\tan x}{1+\sin^2 x}}$, $0\le x\le \frac{\pi}{4}$.

F. Improper integrals.

1) Use the trigonometric identity $\sin(2x)=2\sin x\cos x$ along with $\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}$ to show that $\int_0^\infty \frac{\sin x\cos x}{x}\,dx=\frac{\pi}{4}$.

2) Use integration by parts in 1) to derive the formula $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx=\frac{\pi}{2}$.

3) Use the trigonometric identity $\cos^2 x+\sin^2 x=1$ along with 2) to obtain $\int_0^\infty \frac{\sin^4 x}{x^2}\,dx=\frac{\pi}{4}$.

4) Use two integration by parts, and the result of 3) to obtain $\int_0^\infty \frac{\sin^4 x}{x^4}\,dx=\frac{\pi}{3}$.

5) Let $f:[0,\infty)\longrightarrow[0,\infty)$ be a continuous nonincreasing function such that $\lim_{x\to\infty} f(x)=0$ and $\int_0^x f(t)\,dt-xf(x)\le M$, for all $x\ge 0$, for some $M>0$. Show that the improper integral $\int_0^\infty f(t)\,dt$ converges.

6) Set $I=\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx$ and $J=\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx$.
   i) Show that the improper integral converges.
   ii) Show that $I=J$.
   iii) Show that $I+J=-\pi\ln 2$, and derive the value of I.

G. Sequences and Series.

1) Consider the sequence (a_n) given by $a_1=e$ and $a_{n+1}=\frac{e^n}{a_n}$, $\quad n=1,\ 2,...$

   We want to evaluate $S=\sum_{n=1}^{\infty}\frac{a_n}{e^n}=\sum_{n=1}^{\infty}\frac{1}{a_{n+1}}$.

   i) Show that for each $p\in\mathbb{N}$, one has: $a_{2p+2}=e^p$ and $a_{2p+1}=e^{p+1}$.

   ii) Derive from i) that $S=\frac{e+e^{-1}}{e-1}$.

2) Consider the sequence (u_n) defined by $u_0>0$ and $u_{n+1}=u_ne^{-u_n}$, $\quad n=1,\ 2,....$

i) Show that $\lim_{n\to\infty} u_n = 0$.

ii) Show that $\sum_{n=1}^{\infty} u_n = \infty$.

3) Let (a_n) be a sequence of real numbers that converges to some real number a. For each $n \geq 1$, set $v_n = \frac{a_1+a_2+...+a_n}{n}$ and $w_n = \frac{a_1^2+a_2^2+...+a_n^2}{n}$.

i) Show that $\lim_{n\to\infty} v_n = a$ and $\lim_{n\to\infty} w_n = a^2$.

ii) Find $\lim_{n\to\infty} \frac{2}{n^2} \sum_{1\leq k<l\leq n} a_k a_l$.

4) Determine for which values of the positive parameters the given series converges or diverges.

i) $\sum_{n=0}^{\infty} \frac{a^{b^n}}{b^{a^n}}$, ii) $\sum_{n=1}^{\infty} \frac{n!}{n^{nk}}$, iii) $\sum_{m=1}^{\infty} \left(\frac{pm+q}{m+r}\right)^{m \ln m}$.

5) Let λ be a nonzero real number. with $|\lambda| < 1$, and for each nonnegative integer n, set

$$u_n = \int_0^{2\pi} \frac{\cos(nx)}{1 - 2\lambda \cos x + \lambda^2}\, dx.$$

i) Given that $1 - 2\lambda \cos x + \lambda^2 = (1 - \lambda e^{ix})(1 - \lambda e^{-ix})$, show that $u_n = \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \lambda^{m+p} \int_0^{2\pi} \cos(nx) e^{i(m-p)x} dx$.

ii) Derive from i) that $u_n = \frac{\pi \lambda^n}{1 - \lambda^2}$.

6) Let (u_n) be a sequence of positive real numbers. For each integer $n \geq 0$, set $S_n = \sum_{p=0}^{n} u_p$, and assume that $\lim_{n\to\infty} \frac{S_n}{nu_n} = \alpha$, for some $\alpha > 0$.

i) Determine whether the series $\sum_{n=0}^{\infty} u_n$ converges or diverges. (You may use a contradiction argument.)

ii) For each $n \geq 1$, set $a_n = nS_n - (n-1)S_{n-1}$ and $b_n = nu_n$. a) Find $\lim_{n\to\infty} \frac{a_n}{b_n}$.

b) Derive from a), $\lim_{n\to\infty} \frac{a_1 + a_2 + ... + a_n}{b_1 + b_2 + ... + b_n}$.

iii) Derive from i) and ii), $\lim_{n\to\infty} \frac{u_1 + 2u_2 + ... + nu_n}{n^2 u_n}$.

7) Let $p > 0$. Find $\lim_{n\to\infty} u_n$ if $u_n = \sum_{k=1}^{n} \frac{k^{np}}{n^{np}}$.