Stabilization of the wave equation with localized damping: old results, new results

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Overview

- The frictional damping case

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- The wave equation with Kelvin-Voigt damping
  - Well-posedness and strong stability
  - Polynomial stability
  - Exponential stability
- Some extensions and open problems

The linear damping case

$\Omega \subset \mathbb{R}^N$ throughout. Consider the damped wave equation

$$\begin{array}{l} y_{tt} + Ay + ay_t = 0 \text{ in } \Omega \times (0, \infty) \\ y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega), \end{array}$$

where A is a second order elliptic operator with smooth coefficients, and a is smooth. In general $a(x) \geq a_0 > 0$ a.e. in the damping region ω and $a(x) \equiv 0$ in $\Omega \setminus \omega$, for some positive constant a_0.

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Introduce the energy

$$E(t) = \frac{1}{2} \int_\Omega \{|y_t(x,t)|^2 + |A^{\frac{1}{2}} y(x,t)|^2\}\, dx, \quad \forall t \geq 0.$$

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- Rauch-Taylor, 1974: Ω is a compact manifold without boundary, use microlocal analysis to prove the exponential decay of the energy, provided there exists $T > 0$:
  every ray of geometric optics meets $\omega \times (0, T)$.

The linear damping case

- Haraux, 1989: $A = -\Delta$, Ω is a bounded open set with smooth boundary, Dirichlet boundary conditions are imposed on the boundary, and ω is a neighborhood of a portion of the boundary, uses observability to derive the exponential decay of the energy.

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- Zuazua, 1990-1991: considers the semilinear wave equation $y_{tt} - \Delta y + f(y) + ay_t = 0$, Ω is a bounded open set with smooth boundary, Dirichlet boundary conditions are imposed on the boundary, and ω is a neighborhood of the whole boundary. Proves exponential decay of the energy using multipliers technique and compactness-uniqueness. Does same for some unbounded domains.

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- Nakao, 1996: linear wave equation, $a \in C^m(\bar{\Omega})$, $m > N/2$, $\exists 0 < p < 1 : \int_\omega \frac{dx}{a(x)^p} < \infty$, proves $E(t) = O((1+t)^{\frac{-2mp}{N}})$ for smooth initial data. Discusses a semilinear version with $m = 4$.

The linear damping case

- Lebeau, 1996: Ω is a compact manifold with C^∞ boundary and the damping coefficient $a \in C^\infty(\bar{\Omega})$, uses microlocal analysis to show that $E(t) = O(\frac{\log(3+\log(3+t))}{\log(3+t)})$ for smooth initial data when the damping region is an arbitrary nonempty open set.
- Liu, 1997: introduces the notion of piecewise multipliers, and shows that for the wave equation with Dirichlet boundary conditions, more general feedback control regions may be built; in particular, if Ω is a spherical ball, the damping region may be chosen to be a neighborhood of a diameter, and in 2D, if Ω is a rectangular region, the damping region could be a neighborhood of a diagonal.

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  - $a \in L^r(\Omega)$, with $r \geq \frac{3N+\sqrt{9N^2-16N}}{4}$ if $N \geq 3$, $E(t) = O((1+t)^{\frac{-2r(r-2)}{N(3r-2)}})$.

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  - Those polynomial decay results were improved in 2007 and 2006 respectively in the framework of the dynamic elasticity equations.
  - exponential decay of the energy when $a \in L^\infty(\Omega)$ with $1/a \in L^\infty(\omega)$.

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- Many other works follow...

The nonlinear damping case

Consider the nonlinearly damped wave equation

$$\begin{aligned} & y_{tt} - \Delta y + ag(y_t) = 0 \text{ in } \Omega \times (0, \infty) \\ & y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega), \end{aligned}$$

where $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and nondecreasing.

- Dafermos, 1978: Uses the Lasalle invariance principle to show that the energy decays to zero when the damping region is an arbitrary nonempty open set, provided g is continuously differentiable with a bounded derivative and strictly increasing; no decay estimate is provided.

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- Haraux, 1985: Improves Dafermos' result to include nonlinearities g that are neither smooth nor strictly increasing, but which do possess a monotone graph. No decay rate.

The nonlinear damping case

- Slemrod, 1989: Considers the wave equation $y_{tt} - \Delta y + ag(y_t, \nabla y) = 0$ in $\Omega \times (0, \infty)$.
  Drops the monotonicity hypothesis and replaces it with the assumption that g be globally Lipschitz, and have its graph in the first and third quadrants. Shows the weak decay of solutions to zero in the energy space. He introduces a weak notion of invariance principle through the use of Young measures.

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- Nakao, 1996 nonlinearity has the usual polynomial growth, several energy decay estimates (exponential, polynomial) are provided by combining the multipliers technique, Nakao's difference inequalities, and a compactness uniqueness argument.

The nonlinear damping case

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- Martinez, 1998 (Thesis): Uses his new integral estimate to prove how the energy decay rate depends precisely on the nonlinearity g; in general, $E(t) = O((G^{-1}(1/t))^2)$ with $G(s) = sg(s)$. But, in particular, if $\frac{g(s)}{s}$ vanishes at zero and is increasing on some interval $[0, r]$ for some $r > 0$, then $E(t) = O((g^{-1}(1/t))^2)$.

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- Vancostenoble, 1998: Improves the work of Slemrod to include nonlinearities $g(s, p)$ that grow polynomially in s and are globally Lipschitz in p.

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- Alabau, 2005: Improves the integral inequality of Martinez to allow for more general decay rates.
- Bellassoued, 2005: Improves the logarithmic decay rate of Lebeau for arbitrary damping domains to include polynomially growing nonlinearities; in particular, he shows $E(t) = O((\log(2+t))^{-1})$.

The nonlinear damping case

- Lasiecka-Toundykov, 2006: Drop the requirement that the nonlinearity have a linear growth at infinity from the works of Martinez and Alabau. Discuss wave equations with source terms.

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- Ammari-Alabau: Show that the stabilization of the wave equation with a nonlinear damping can be derived from the stabilization of the wave equation with linear damping, thereby improving earlier results by allowing for damping regions satisfying the geometric optics condition of Rauch-Taylor.
- Many more works by others...

The one-dimensional problem

- Liu-Liu, 1998: Consider the wave equation

$$
\begin{aligned}
&y_{tt} - (py_x)_x - (D_a\, y_{tx})_x = 0 \text{ in } (0, L) \times (0, \infty) \\
&y(0, t) = y(L, t) = 0 \text{ on } (0, \infty), \\
&y(0) = y^0 \in H_0^1(0, L), \quad y_t(0) = y^1 \in L^2(0, L).
\end{aligned}
$$

Show that the exponential decay of the energy fails for piecewise constant coefficients. No decay rate.

- Renardy, 2004: Consider the wave equation

$$\begin{array}{l} y_{tt} - (y_x)_x - (b\, y_{tx})_x = 0 \text{ in } (-1,1) \times (0,\infty) \\ y(-1,t) = y(1,t) = 0 \text{ on } (0,\infty), \\ y(0) = y^0 \in H_0^1(0,L), \quad y_t(0) = y^1 \in L^2(0,L). \end{array}$$

Shows that if

$$\begin{array}{l} b \in C^1([-1,1]), \quad b(x) = 0 \text{ on } [-1,0], \quad b(x) > 0 \text{ on } (0,1] \\ \lim\limits_{x \to 0^+} \dfrac{b'(x)}{x^\alpha} = k > 0, \text{ for some } \alpha > 0, \end{array}$$

the sequence of the corresponding eigenvalues $\{\lambda_n\}$ satisfies $\Re\lambda_n \to -\infty$ as $n \to \infty$.

The multidimensional problem

Consider the wave equation with localized Kelvin-Voigt damping

$$\rho(x)y_{tt} - \mathrm{div}(a\nabla y) - div(b\nabla y_t) = 0 \text{ in } \Omega \times (0,\infty)$$
$$y = 0 \text{ on } \Sigma = \Gamma \times (0,\infty), \quad y(0) = y^0 \in H_0^1(\Omega), \quad y_t(0) = y^1 \in L^2(\Omega).$$

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- Liu-Rao, 2006: Show the exponential decay of the energy provided

  $$\rho,\ a,\ b \in C^{1,1}(\bar{\Omega}),\ \Delta b \in L^\infty(\Omega),$$
  $$\exists C > 0 : |\nabla b(x)|^2 \leq Cb(x) \text{ a.e. in } \omega.$$

  and ω is a neighborhood of the whole boundary. They rely on the frequency domain method (FDM) combined with multipliers technique and mollifiers.

The multidimensional problem

- Tebou, 2012: Improves the energy exponential decay result of Liu-Rao by dropping the hypothesis $\Delta b \in L^\infty(\Omega)$, and by allowing for a more general class of damping regions. Proves a polynomial energy decay estimate when the damping coefficient is bounded measurable only. Relies on FDM, piecewise multipliers of Liu, and auxiliary variables.

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  - the feedback control support $\omega = \{x \in \Omega; a(x) > 0\}$ satisfies the same geometric control condition as in the work of Rauch-Taylor,

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$$|\partial^\alpha a(x)| \leq C_\alpha a^{\frac{k-|\alpha|}{k}}, \quad |\alpha| \leq 2,$$

  for some $k > 2$, and

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    $$|\partial^\alpha a(x)| \leq C_\alpha a^{\frac{k-|\alpha|}{k}}, \quad |\alpha| \leq 2,$$

    for some $k > 2$, and
  - the initial displacement is identically zero in Ω.

The multidimensional problem

- Tebou, 2016: Extends the results obtained for the wave equation to the elasticity equations.
- Ammari-Hassine-Robbiano, 2018: $b(x) = d1_\omega$, $d > 0$ is a constant, and ω is an arbitrary nonempty open set in Ω; they use microlocal analysis to prove the logarithmic decay of the energy.

Problem formulation

Consider the wave equation with localized Kelvin-Voigt damping

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&y_{tt} - \Delta y - div(a\nabla y_t) = 0 \text{ in } \Omega \times (0, \infty)\\
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Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
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where
Ω= bounded domain in $\mathbb{R}^N$, $N \geq 1$,
Γ= boundary of Ω is smooth.

- The damping coefficient is nonnegative, bounded measurable, and is positive in a nonempty open subset ω of Ω,
- the system may be viewed as a model of interaction between an elastic material (portion of Ω where $a \equiv 0$), and a viscoelastic material (portion of Ω where $a > 0$).

Remark

If $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, then the system is well-posed in $H_0^1(\Omega) \times L^2(\Omega)$. Introduce the energy

$$E(t) = \frac{1}{2}\int_\Omega \{|y_t(x,t)|^2 + |\nabla y(x,t)|^2\}\, dx, \quad \forall t \geq 0.$$

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We have the dissipation law:

$$\frac{dE}{dt}(t) = -\int_\Omega a(x)|\nabla y_t(x,t)|^2\,dx \text{ a.e. } t > 0.$$

The energy is a nonincreasing function of the time variable.

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Question 1: Does the energy approach zero?

Question 2: When the energy does go to zero, how fast is its decay, and under what conditions?

Introduce the Hilbert space over the field $\mathbb{C}$ of complex numbers $\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega)$, equipped with the norm

$$||Z||_{\mathcal{H}}^2 = \int_\Omega \{|v|^2 + |\nabla u|^2\}\, dx, \quad \forall Z = (u, v) \in \mathcal{H}.$$

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the unbounded operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & div(a\nabla .) \end{pmatrix}$$

with $D(\mathcal{A}) = \{(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega); \Delta u + div(a\nabla v) \in L^2(\Omega)\}$.

Now if $(y^0, y^1) \in H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$ then it can be shown that the unique solution of the system satisfies

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Note the discrepancy between the regularity of the initial state of the system and that of all other states as the system evolves with time.

This is what makes the stabilization problem at hand trickier than the case of a viscous damping ay_t, or more generally $ag(y_t)$ for an appropriate nonlinear function g.

Theorem 1 [Liu-Rao, 2006]

Suppose that ω is an arbitrary nonempty open set in Ω. Let the damping coefficient a be nonnegative, bounded measurable, and positive in ω. The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$, which is strongly stable:

$$\lim_{t\to\infty} ||S(t)Z^0||_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}.$$

For the sequel we need the geometric constraint (GC) on the subset ω where the dissipation is effective.

(GC). There exist open sets $\Omega_j \subset \Omega$ with piecewise smooth boundary $\partial\Omega_j$, and points $x_0^j \in \mathbb{R}^N$, $j = 1,\ 2, ..., J$, such that $\Omega_i \cap \Omega_j = \emptyset$, for any $1 \leq i < j \leq J$, and:

$$\Omega \cap \mathcal{N}_\delta \left[\left(\bigcup_{j=1}^{J} \Gamma_j \right) \bigcup \left(\Omega \setminus \bigcup_{j=1}^{J} \Omega_j \right) \right] \subset \omega,$$

for some $\delta > 0$, where $\mathcal{N}_\delta(S) = \displaystyle\bigcup_{x \in S} \{y \in \mathbb{R}^N; |x - y| < \delta\},$ for $S \subset \mathbb{R}^N$,

$\Gamma_j = \left\{ x \in \partial\Omega_j; (x - x_0^j) \cdot \nu^j(x) > 0 \right\}$, ν^j being the unit normal vector pointing into the exterior of Ω_j.

Theorem 2

Suppose that ω satisfies the geometric condition (GC). Let the damping coefficient a be nonnegative, bounded measurable, with $a(x) \geq a_0$ a.e. in ω, for some constant $a_0 > 0$. There exists a positive constant C such that the semigroup $(S(t))_{t\geq 0}$ satisfies:

$$||S(t)Z^0||_{\mathcal{H}} \leq \frac{C||Z^0||_{D(\mathcal{A})}}{\sqrt{1+t}}, \quad \forall Z^0 \in D(\mathcal{A}), \quad \forall t \geq 0.$$

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Remark. The polynomial decay estimate in Theorem 2 is in sharp contrast with what happens in the case of a viscous damping of the form ay_t or $ag(y_t)$ for a nondecreasing globally Lipschitz nonlinearity g; in fact, when (GC) holds, the geometric control condition of Bardos-Lebeau-Rauch (every ray of geometric optics intersects ω in a finite time T_0) is met, and exponential decay of the energy should be expected; this is by now well known:

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- $i\mathbb{R} \subset \rho(\mathcal{A})$, (given by Theorem 1)

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- Apply a theorem of Borichev-Tomilov on polynomial decay of bounded semigroups.

We shall prove that there exists a constant $C > 0$ such that for every $U = (f, g) \in \mathcal{H}$, the element $Z = (ib - \mathcal{A})^{-1} U = (u, v)$ in $D(\mathcal{A})$ satisfies:

$$||Z||_{\mathcal{H}} \leq Cb^2 ||U||_{\mathcal{H}}, \quad \forall b \in \mathbb{R}, \ |b| \geq 1$$

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may be recast as

$$\begin{aligned}
&ibu - v = f \text{ in } \Omega\\
&ibv - \Delta u - \operatorname{div}(a(x)\nabla v) = g \text{ in } \Omega\\
&u = 0, \quad v = 0 \text{ on } \Gamma.
\end{aligned}$$

Introduce the new function $u_1 = u - w$, where $w = G(\mathrm{div}(a\nabla v))$, with $G =$inverse of $-\Delta$ with Dirichlet BCs. One notes $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$, and

$$||w||_{H_0^1(\Omega)} \leq \sqrt{|a|_\infty ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}}, \quad ||u_1||_{H_0^1(\Omega)} \leq ||Z||_{\mathcal{H}} + \sqrt{|a|_\infty ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}}.$$

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The second equation in (3) becomes

$$ibv - \Delta u_1 = g \text{ in } \Omega,$$

from which one derives

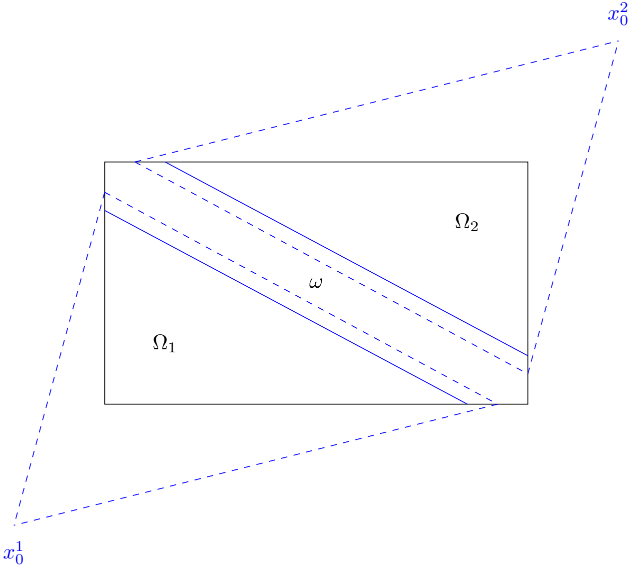
$$|b| ||v||_{H^{-1}(\Omega)} \leq ||u_1||_{H_0^1(\Omega)} + C|g|_2.$$

Let $J \geq 1$ be a an integer. For each $j = 1, 2, ..., J$, set $m^j(x) = x - x_0^j$. Let $0 < \delta_0 < \delta_1 < \delta$, where δ is the same as in the geometric condition stated above. Set

$$S = \left(\bigcup_{j=1}^{J} \Gamma_j \right) \bigcup \left(\Omega \setminus \bigcup_{j=1}^{J} \Omega_j \right),$$
$$Q_0 = \mathcal{N}_{\delta_0}(S), \quad Q_1 = \mathcal{N}_{\delta_1}(S), \quad \omega_1 = \Omega \cap Q_1,$$

and for each j, let φ_j be a function satisfying

$$\varphi_j \in W^{1,\infty}(\Omega), \quad 0 \leq \varphi_j \leq 1, \quad \varphi_j = 1 \text{ in } \bar{\Omega}_j \setminus Q_1, \quad \varphi_j = 0 \text{ in } \Omega \cap Q_0.$$


x_0^2
Ω_2
ω
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The usual multiplier technique leads to the estimate

$$||Z||_{\mathcal{H}}^2 \leq C||U||_{\mathcal{H}}^2 + C|b| \left| \sum_{j=1}^{J} \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{w}\, dx \right|.$$

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Thanks to the estimate on w, one derives

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Combining the two equations, one derives the desired estimate:

$$||Z||_{\mathcal{H}} \leq Cb^2||U||_{\mathcal{H}}.$$

Theorem 3

Suppose that ω satisfies the geometric condition (GC). As for the damping coefficient a, assume

$$a \in W^{1,\infty}(\Omega) \text{ with } |\nabla a(x)|^2 \leq M_0 a(x), \text{ a.e. in } \Omega,$$
$$a(x) \geq a_0 > 0 \text{ a.e. in } \omega_1,$$

for some positive constants M_0 and a_0.
The semigroup $(S(t))_{t\geq 0}$ is exponentially stable; more precisely, there exist positive constants M and λ with

$$||S(t)Z^0||_{\mathcal{H}} \leq M \exp(-\lambda t)||Z^0||_{\mathcal{H}}, \quad \forall Z^0 \in \mathcal{H}.$$

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Remark. In Liu-Rao (2006), the feedback control region ω is a neighborhood of the whole boundary, and the damping coefficient a should further satisfy $\Delta a \in L^\infty(\Omega)$.

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- Apply a theorem due to Prüss or Huang on exponential decay of bounded semigroups.

We shall prove that there exists a constant $C > 0$ such that for every $U = (f, g) \in \mathcal{H}$, the element $Z = (ib - \mathcal{A})^{-1}U = (u, v)$ in $D(\mathcal{A})$ satisfies:

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$$||Z||^2_{\mathcal{H}} \leq C||U||^2_{\mathcal{H}} + C|b| \left| \sum_{j=1}^{J} \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{w} \, dx \right|.$$

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With the smoothness and structural conditions on the coefficient a, it can be shown that, on the one hand

$$C|b| \left| \sum_{j=1}^{J} \int_{\Omega_j} v\varphi_j m^j \cdot \nabla \bar{w}\, dx \right| \leq C|b||\sqrt{a}v|_2(||U||^{\frac{1}{2}}_{\mathcal{H}}||Z||^{\frac{1}{2}}_{\mathcal{H}} + ||Z||_{\mathcal{H}}),$$

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Remark. The following system:

$$\begin{aligned} &y_{tt} - \Delta y - ag(\Delta y_t) = 0 \text{ in } \Omega \times (0, \infty) \\ &y = 0 \text{ on } \Sigma = \Gamma \times (0, \infty), \quad y(0) = y^0, \quad y_t(0) = y^1 \text{ in } \Omega. \end{aligned}$$

But now, the natural the energy space is
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But now, the natural the energy space is $\hat{\mathcal{H}} = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$. When g is globally Lipschitz, and the damping is localized, it has been shown that the energy decays exponentially, by using the Komornik integral inequality and the localized smoothness of solutions. When $a \equiv 1$ in Ω, and $g(s) = |s|^{p-2}s$, it has been established that

$$E(t) \leq \begin{cases} \frac{K_1}{(1+t)^{\frac{3-p}{p-2}}} \text{ if } 2 < p < 3, \\ \frac{K_2}{(\log(2+t))^{\frac{2}{p-2}}} \text{ if } p \geq 3, \end{cases} \qquad \forall t \geq 0.$$

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6. The analogous problem for the plate equation$y_{tt} + \Delta^2 y + \Delta(a\Delta y_t) = 0$ in $\Omega \times (0, \infty)$ with clamped BCs is open in the multidimensional setting. No smoothness on the damping coefficient is needed in the one-dimensional setting, (Liu-Liu, 1998).

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!