

MAC 2313 (Calculus III)
Test 1, Thursday September 28, 2006

Name:

PID:

Remember that no documents or graphing calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. You will not get any credit by just writing down the answer to any of the problems. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.

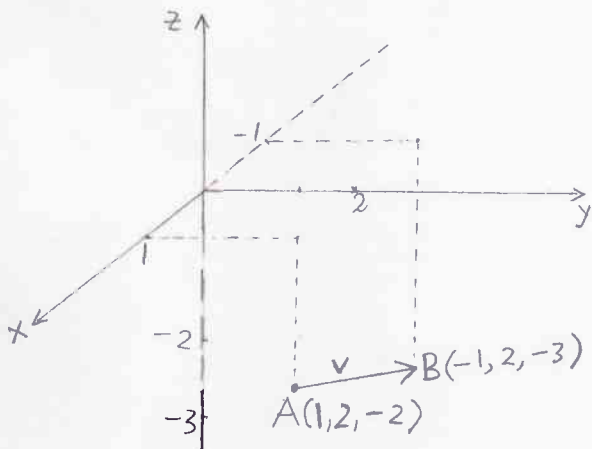
1. [10] Let m be a real number. Consider the surface given by the equation: $x^2 + y^2 + z^2 - 2x + 2my + 4z + 9 = 0$. For which values of m is that surface a sphere? a point?

Eqn of surface may be rewritten $(x-1)^2 + (y+m)^2 + (z+2)^2 = 1+m^2+4-9$
 or $(x-1)^2 + (y+m)^2 + (z+2)^2 = m^2 - 4$ (3)

If $m^2 > 4$, or $m < -2$ or $m > 2$; then the surface is the sphere with center $(1, -m, -2)$ and radius $\sqrt{m^2 - 4}$. (1)

If $m = -2$ or $m = 2$ surface reduces to the point $(1, -m, -2)$.
0.5 0.5

2. [10] Plot the two points $A(1, 2, -2)$, and $B(-1, 2, -3)$, and sketch the vector $\mathbf{v} = \overrightarrow{AB}$. What is $\|\mathbf{v}\|$? Find to the nearest degree the angle that \mathbf{v} makes with the positive x -axis.



$$\mathbf{v} = -2\vec{i} - \vec{k} \quad 2$$

$$\|\mathbf{v}\| = \sqrt{4+1} = \sqrt{5} \quad 2$$

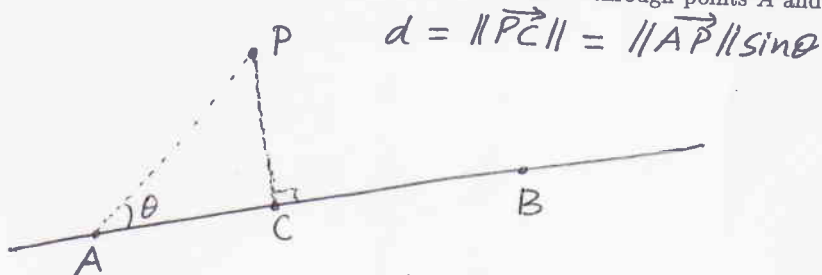
$$\cos\theta = \frac{\mathbf{v} \cdot \vec{i}}{\|\mathbf{v}\| \|\vec{i}\|} = \frac{-2}{\sqrt{5}} \quad 2$$

$$\theta \approx 153^\circ \quad 2$$

3. [10] If $\mathbf{u} = \vec{i} + 2\vec{j} + \vec{k}$, and $\mathbf{b} = -\vec{i} + \vec{j} + 3\vec{k}$, find the component of \mathbf{u} that is parallel to \mathbf{b} , and the component of \mathbf{b} that is parallel to \mathbf{u} .

We may write $\mathbf{u} = \alpha \mathbf{b} + \mathbf{v}$ with \mathbf{v} orthogonal to \mathbf{b} .
 So $\mathbf{u} \cdot \mathbf{b} = \alpha \|\mathbf{b}\|^2 + \mathbf{v} \cdot \mathbf{b}$; so $\alpha = \frac{\mathbf{u} \cdot \mathbf{b}}{\|\mathbf{b}\|^2}$. The required component of $\mathbf{u} \parallel \mathbf{b}$ is $\frac{\mathbf{u} \cdot \mathbf{b}}{\|\mathbf{b}\|^2} \mathbf{b} = \frac{-1+2+3}{1+1+9} \mathbf{b} = \frac{4}{11} (-\vec{i} + \vec{j} + 3\vec{k})$
 Similarly, the component of $\mathbf{b} \parallel \mathbf{u}$ is $\frac{\mathbf{b} \cdot \mathbf{u}}{\|\mathbf{u}\|^2} \mathbf{u} = \frac{4}{1+4+1} \mathbf{u} = \frac{2}{3} (\vec{i} + 2\vec{j} + \vec{k})$

4. [10] Show that in 3-space the distance d from a point P to the line L through points A and B can be expressed as $d = \frac{\|\vec{AP} \times \vec{AB}\|}{\|\vec{AB}\|}$.



Now $\|\vec{AP} \times \vec{AB}\| = \|\vec{AP}\| \|\vec{AB}\| \sin \theta$; so $\sin \theta = \frac{\|\vec{AP} \times \vec{AB}\|}{\|\vec{AP}\| \|\vec{AB}\|}$
 Hence $d = \|\vec{AP}\| \cdot \frac{\|\vec{AP} \times \vec{AB}\|}{\|\vec{AP}\| \|\vec{AB}\|} = \frac{\|\vec{AP} \times \vec{AB}\|}{\|\vec{AB}\|}$.

5. [10] If θ is the angle between $u = -2\vec{i} + \vec{j} + \vec{k}$, and $v = \vec{i} - 2\vec{j} + \vec{k}$, find $\sin \theta$. Is θ acute or obtuse?

$\|u \times v\| = \|u\| \|v\| \sin \theta$; so $\sin \theta = \frac{\|u \times v\|}{\|u\| \|v\|}$
 Now $u \times v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 3\vec{i} + 3\vec{j} + 3\vec{k}$

So $\sin \theta = \frac{\sqrt{9+9+9}}{\sqrt{4+1+1} \cdot \sqrt{1+4+1}} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2}$.

The sign of $u \cdot v$ will tell us whether θ is acute or obtuse.

Now $u \cdot v = -2 - 2 + 1 = -3 < 0$; so θ is obtuse.

6. [10] Show that the two lines $L_1: x = 1 - t, y = -1, z = t$, and $L_2: x = 1, y = t, z = 2 + 2t$ intersect, and find their point of intersection.

Do there exist t_1, t_2 such that

$1 - t_1 = 1 \rightarrow t_1 = 0$
 $-1 = t_2 \rightarrow t_2 = -1$ } reporting these values in last eqn yields
 $t_1 = 2 + 2t_2$? $0 = 2 + 2(-1)$ which is true

So L_1 and L_2 intersect at $(1, -1, 0)$.

7. [12] a) Convert $(3/2, \sqrt{3}/2, -1)$ from rectangular coordinates to spherical coordinates. b) Convert the equation $x^2 - y^2 - z^2 = 1$ from rectangular coordinates to cylindrical coordinates.

a) $\rho = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + (-1)^2} = \sqrt{\frac{9}{4} + \frac{3}{4} + 1} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{\sqrt{3}/2}{3/2}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{3}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$\phi = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

b) $x = r\cos\theta$ $y = r\sin\theta$, $z = z$

So eqn becomes $r^2\cos^2\theta - r^2\sin^2\theta - z^2 = 1$, or $r^2(\cos^2\theta - \sin^2\theta) - z^2 = 1$

or $\underline{\underline{r^2\cos(2\theta) = z^2 + 1}}$

8. [20] If $r(t) = e^{-t}\vec{i} + \sin(1-e^{-t})\vec{j} + \cos(1-e^{-t})\vec{k}$, find T, N, B, as well as the equations of the osculating, and the rectifying planes at time $t = 0$.

$r'(t) = -e^{-t}\vec{i} + e^{-t}\cos(1-e^{-t})\vec{j} - e^{-t}\sin(1-e^{-t})\vec{k}$

$\|r'(t)\| = e^{-t}\sqrt{1 + \cos^2(1-e^{-t}) + \sin^2(1-e^{-t})} = \sqrt{2}e^{-t}$

$T(0) = r'(0)/\|r'(0)\| = \frac{-\vec{i} + \vec{j}}{\sqrt{2}}$. $T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{-\vec{i} + \cos(1-e^{-t})\vec{j} - \sin(1-e^{-t})\vec{k}}{\sqrt{2}}$

$T'(t) = \frac{-e^{-t}\sin(1-e^{-t})\vec{j} - e^{-t}\cos(1-e^{-t})\vec{k}}{\sqrt{2}}$

$N(0) = \frac{T'(0)}{\|T'(0)\|} = \frac{-\vec{k}}{\sqrt{2}/\sqrt{2}} = -\vec{k}$, $B(0) = T(0) \times N(0) = \frac{(-\vec{i} + \vec{j}) \times (-\vec{k})}{\sqrt{2}}$

$= \frac{\vec{i} \times \vec{k} - \vec{j} \times \vec{k}}{\sqrt{2}} = \frac{-\vec{j} - \vec{i}}{\sqrt{2}}$

Planes pass through $r(0) = (1, 0, 1)$

Eqn of osc. plane: $-(x-1) - y = 0$ or $x + y - 1 = 0$

Eqn of rect. plane: $-(z-1) = 0$ or $\underline{\underline{z = 1}}$

9. [10] If $r(t) = \sqrt{3}e^t\vec{i} + \sin(e^t)\vec{j} + \cos(e^t)\vec{k}$, find an arc length parametrization of the curve that has the same orientation as the given curve, and has $t = 0$ as the reference point.

arc length parameter is $s = \int_0^t \|r'(u)\| du = \int_0^t \sqrt{3e^{2u} + e^{2u}\cos^2(e^u) + e^{2u}\sin^2(e^u)} du$

$e^t = \frac{s+2}{2}$

$= 2 \int_0^t e^u du$
 $= 2e^u \Big|_0^t = 2(e^t - 1)$

$r(s) = \frac{\sqrt{3}}{2}(s+2)\vec{i} + \sin\left(\frac{s+2}{2}\right)\vec{j} + \cos\left(\frac{s+2}{2}\right)\vec{k}$