MAC 2311 (Calculus I)— Answers
TEST 1, Wednesday February 4, 2015

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good Luck.

1. [30] Evaluate the following limits (Show all your work. You will not get any credit(s) by guessing the correct answer(s). You cannot use de l'Hopital's rule for any of the limits, otherwise you'll get a zero. If a limit is infinite, clearly state whether it is $+\infty$ or $-\infty$.)

a) $\lim_{x\to -1} \frac{2x^3-x^2-3}{5x^2-4x+1} = \frac{2(-1)^3-(-1)^2-3}{5(-1)^2-4(-1)+1}$
$= \frac{-2-1-3}{5+4+1}$
$= -\frac{6}{10} = -\frac{3}{5}$

b) $\lim_{x\to -\infty} \frac{-5x^4+3x+7^{2014}}{9^{2016}-11^{99}x^2+3x^3} = \lim_{x\to -\infty} \frac{-5x^4}{3x^3}$
$= -\frac{5}{3}\lim_{x\to -\infty} x$, as $\frac{x^4}{x^3}=x$
$= -\frac{5}{3}(-\infty) = +\infty$

c) $\lim_{x\to 3^-} \frac{1-x-x^2}{3-x} = \lim_{x\to 3^-}(1-x-x^2)\cdot \lim_{x\to 3^-}\frac{1}{-(x-3)}$
$=(1-3-3^2)\cdot(-\lim_{x\to 3^-}\frac{1}{x-3})$
$= -11\cdot(-(-\infty))$
$= -11(+\infty)$
$= -\infty$

d) $\lim_{x\to 1} \frac{3-\sqrt{10x-1}}{x-1} = \lim_{x\to 1} \frac{(3-\sqrt{10x-1})(3+\sqrt{10x-1})}{(x-1)(3+\sqrt{10x-1})}$
$= \lim_{x\to 1} \frac{9-(10x-1)}{(x-1)(3+\sqrt{10x-1})}$
$= \lim_{x\to 1} \frac{10(1-x)}{(x-1)(3+\sqrt{10x-1})}$
$= \lim_{x\to 1} \frac{-10\cancel{(x-1)}}{\cancel{(x-1)}(3+\sqrt{10x-1})}$
$= \frac{-10}{3+\sqrt{9}}$
$= -\frac{10}{6}$
$= -\frac{5}{3}$

e) $\lim_{x\to 0} \frac{\sin(5x)}{\sin(15x)} = \lim_{x\to 0} \frac{\frac{\sin(5x)}{5x}\cdot 5\cancel{x}}{\frac{\sin(15x)}{15x}\cdot 15\cancel{x}}$
$= \frac{5}{15}\cdot \frac{\lim_{x\to 0}\frac{\sin(5x)}{5x}}{\lim_{x\to 0}\frac{\sin(15x)}{15x}} = \frac{1}{3}\cdot\frac{1}{1} = \frac{1}{3}$

f) $\lim_{x\to 2^+} \frac{8-5x}{x^2+4} = \frac{8-5(2)}{2^2+4}$
$= \frac{-2}{8}$
$= -\frac{1}{4}$

g) $\lim_{x\to -2} \frac{x^3+8}{x^2+x-2} = \lim_{x\to -2} \frac{\cancel{(x+2)}(x^2-2x+4)}{\cancel{(x+2)}(x-1)}$
$= \lim_{x\to -2}\frac{x^2-2x+4}{x-1}$
$= \frac{(-2)^2-2(-2)+4}{-2-1}$
$= \frac{4+4+4}{-3} = \frac{12}{-3} = -4$

h) $\lim_{x\to +\infty} \sqrt{x^2-5x+6}-x = \lim_{x\to+\infty}\frac{(\sqrt{x^2-5x+6}-x)(\sqrt{x^2-5x+6}+x)}{\sqrt{x^2-5x+6}+x}$
$= \lim_{x\to+\infty}\frac{\cancel{x^2}-5x+6-\cancel{x^2}}{\sqrt{x^2-5x+6}+x}$
$= \lim_{x\to+\infty}\frac{-5x}{\sqrt{x^2}+x} = \lim_{x\to+\infty}\frac{-5x}{x+x}$
$= \lim_{x\to+\infty}\frac{-5\cancel{x}}{2\cancel{x}} = -\frac{5}{2}$

i) $\lim_{x\to -\infty} \sin(\frac{\pi x^2-3x+7}{3x^2+24x-10^{2015}}) = \sin(\lim_{x\to-\infty}\frac{\pi x^2-3x+7}{3x^2+24x-10^{2015}})$
$= \sin(\lim_{x\to-\infty}\frac{\pi \cancel{x^2}}{3\cancel{x^2}})$
$= \sin(\frac{\pi}{3})$
$= \frac{\sqrt{3}}{2}$

j) $\lim_{x\to \frac{\pi}{4}} \frac{\sin x}{x} = \frac{\sin\frac{\pi}{4}}{\frac{\pi}{4}} = \frac{4}{\pi}\cdot\frac{\sqrt{2}}{2} = \frac{2\sqrt{2}}{\pi}$

2. [10, **Bonus**] Find all values of x at which the given function is continuous. You must carefully explain your answer to get any credits. a) $f(x) = \log_{5x-1}(3-2x)$ $= \dfrac{\ln(3-2x)}{\ln(5x-1)}$. f is continuous for

$3-2x>0$ and $5x-1>0$, and $5x-1 \neq 1$

$3>2x$ or $x<\frac{3}{2}$ $\quad$ $x>\frac{1}{5}$ $\quad$ $x \neq \frac{2}{5}$.

Hence f is continuous on $(\frac{1}{5}, \frac{2}{5}) \cup (\frac{2}{5}, \frac{3}{2})$

b) $g(x) = \arccos(e^x)$, arccos is continuous on $[-1,1]$, $e^x>0$, and $x \mapsto e^x$ is continuous everywhere. So g is continuous for $e^x \leq 1$, or $x \leq \ln 1 = 0$

g is continuous on $(-\infty, 0]$.

Note: if $x>0$, then $e^x>1$, and $\arccos(e^x)$ is not defined.

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3. [10] a) Write down the rigorous definition of $\lim_{x\to x_0} f(x) = L$. b) Use the rigorous definition of limit to prove that $\lim_{x\to -7}(4-3x) = 25$.

a) For every $\varepsilon>0$, there exists $\delta>0$: $0<|x-x_0|<\delta \Rightarrow |f(x)-L|<\varepsilon$.

b) Let $\varepsilon>0$, and find $\delta>0$: $0<|x+7|<\delta \Rightarrow |4-3x-25|<\varepsilon$.

For $|-3x-21|<\varepsilon$, it is enough that $|-3(x+7)|<\varepsilon$

— " — " — " $|-3||x+7|<\varepsilon$

— " — " — " — " $3|x+7|<\varepsilon$

— " — " — " $|x+7|<\varepsilon/3$

We may choose $\delta = \varepsilon/3$.

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4. [10] a) State the intermediate value theorem. b) Use it to show that the equation $x + e^x = 0$ has a solution in the open interval $(-1, 0)$.

a) Let f be continuous on the closed interval $[a,b]$. Suppose that $f(a)\cdot f(b)<0$. There exists at least one x_0 in the open interval (a,b) with $f(x_0)=0$.

b) Set $f(x) = x+e^x$. Then f is continuous on $[-1,0]$ as the sum of two functions that are continuous everywhere.

$f(-1)\cdot f(0) = (-1+e^{-1})(0+1) = -1+\frac{1}{e} = \frac{1-e}{e} < 0$

All the requirements of the IVT are met; so there exist at least one solution to $f(x)=0$ on $(-1,0)$.