MAP 2302 (Differential Equations) — Answers
TEST 1, Friday February 19, 2016

Name: PID:

Remember that no documents or calculators are allowed during the test. You must show all your work to deserve the full credit assigned to any question. 4 pages. Total =85 points with 5 points bonus.

1. [15] Solve the separable equation: $(e^{-s}+2)\cos t dt + e^{-s}(1+\sin^2 t)ds = 0$.

$$\frac{\cos t}{1+\sin^2 t}dt + \frac{e^{-s}}{e^{-s}+2}ds = 0$$

Integrate:

$$\int \frac{\cos t}{1+\sin^2 t}dt + \int \frac{e^{-s}}{e^{-s}+2}ds = c, \quad c = \text{arbitrary constant}$$

$u = \sin t$, $du = \cos t\,dt$; $v = e^{-s}+2$, $dv = -e^{-s}ds$

$$\int \frac{du}{1+u^2} + \int -\frac{dv}{v} = c$$

$$\arctan(u) - \ln(v) = c$$

$$\arctan(\sin t) - \ln(2+e^{-s}) = c.$$

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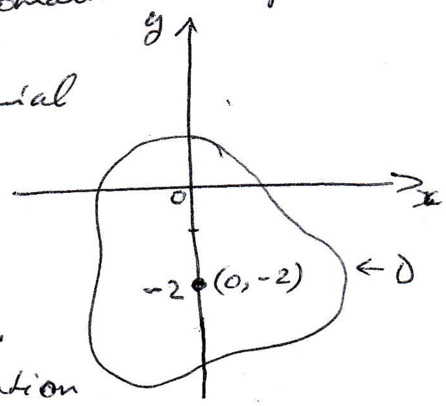
2. [10] State Theorem 1.1 from the text. Use that theorem to show that the initial-value problem:

$$\begin{cases} \dfrac{dy}{dx} = x^2y^3 - x^3y^2 \\ y(0) = -2. \end{cases}$$

has a unique solution defined on some sufficiently small interval $|x| \le h$ about $x_0 = 0$.

Statement: See text or notes

Set $f(x,y) = x^2y^3 - x^3y^2$. Let D be a domain in the plane containing the point $(0,-2)$.



- f is continuous on D, as f is a polynomial in x and y
- $\frac{\partial f}{\partial y}(x,y) = 3x^2y^2 - 2x^3y$; so $\frac{\partial f}{\partial y}$ is also continuous on D.

All the hypotheses of Theorem 1.1 are met.
So the given IVP has a unique solution defined on some small enough interval $|x| \le h$ about $x_0 = 0$.

3. [20] a) Reduce the equation $(5x - 4y + 6)dx + (10x - 8y - 3)dy = 0$ to a separable equation. Do not solve the separable equation obtained.

$a_1 = 5, \; b_1 = -4 \quad a_2 = 10, \; b_2 = -8; \; a_1 b_2 = -40 = b_1 a_2$

Set $z = 5x - 4y$; then $y = \frac{5x - z}{4}$; $dy = \frac{5}{4}dx - \frac{dz}{4}$

D.E becomes

$$(z+6)dx + (2z-3)\left(\frac{5}{4}dx - \frac{dz}{4}\right) = 0$$

or $4(z+6)dx + 5(2z-3)dx - (2z-3)dz = 0$

or $(14z+9)dx - (2z-3)dz = 0$, which is a separable equation

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b) Find all values of n such that $f(x) = e^{nx}$ solves the differential equation: $y''' - y'' - 2y' + 2y = 0$. Do not solve the differential equation.

$f'(x) = ne^{nx}, \; f''(x) = n^2 e^{nx}, \; f'''(x) = n^3 e^{nx}$

Reporting those in the D.E, one finds

$(n^3 - n^2 - 2n + 2)e^{nx} = 0$ for all x; so

$n^3 - n^2 - 2n + 2 = 0$

$n^2(n-1) - 2(n-1) = 0$

$(n^2 - 2)(n-1) = 0$

$n^2 = 2 \rightarrow n = \pm\sqrt{2}$, and $n - 1 = 0 \rightarrow n = 1$

$n_1 = -\sqrt{2}, \; n_2 = \sqrt{2}, \; n_3 = 1$

4. [20] Find the orthogonal trajectories to the family of curves $y^2 = 2x - 1 + ce^{-2x}$.

Find D.E for the given family

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(2x-1+ce^{-2x})$$

$$2y\frac{dy}{dx} = 2 - 2ce^{-2x}$$

$$\frac{dy}{dx} = \frac{1-ce^{-2x}}{y}$$

O.T. D.E.: $\frac{dy}{dx} = \frac{-y}{1-ce^{-2x}}$. Now $1-ce^{-2x} = 2x-y^2$

$$= \frac{-y}{2x-y^2}$$

or

$$\underbrace{y}_{M(x,y)}\,dx + \underbrace{(2x-y^2)}_{N(x,y)}\,dy = 0$$

$\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y) = 2-1=1 \neq 0$; D.E is not exact

$$\frac{\frac{\partial N}{\partial x}(x,y) - \frac{\partial M}{\partial y}(x,y)}{M} = \frac{1}{y}$$

Integrating factor: $\mu(y) = e^{\int \frac{1}{y}dy} = e^{\ln y} = y$

New D.E

$y^2 dx + (2xy - y^3)dy = 0$, is exact

using grouping:

$$\underbrace{y^2dx + y(2x)dy}_{y^2d(x)+xd(y^2)} - y^3dy = 0$$

$$y^2d(x)+xd(y^2) - d\left(\frac{y^4}{4}\right) = 0$$

$$d(y^2x) - d\left(\frac{y^4}{4}\right) = 0$$

$$d\left(y^2x - \frac{y^4}{4}\right) = 0$$

$y^2x - \frac{y^4}{4} = c$, c = arbitrary constant

are the orthogonal trajectories.

5. [20] Solve the Bernoulli initial-value problem: $\frac{dy}{dx} + y = \frac{2x}{y}$, $y(0) = 1$.

multiply D.E by y:

$$y\frac{dy}{dx} + y^2 = 2x$$

Set $z = y^2$, then $\frac{dz}{dx} = 2y\frac{dy}{dx}$

D.E becomes

$$\frac{1}{2}\frac{dz}{dx} + z = 2x \quad \text{or} \quad \frac{dz}{dx} + 2z = 4x, \text{ which is linear}$$

The solution formula yields

$$z = e^{\int -2dx}\left[\int 4xe^{\int 2dx}dx + c\right], \quad c = \text{arbitrary constant}$$

$$= e^{-2x}\left[\int 4xe^{2x}dx + c\right]$$

$$= e^{-2x}\left[2xe^{2x} - \int 2e^{2x}dx + c\right]$$

$$= e^{-2x}\left[2xe^{2x} - e^{2x} + c\right]$$

$$= 2x - 1 + ce^{-2x}$$

$$y^2 = 2x - 1 + ce^{-2x} = \text{general soln}$$

Soln of IVP:

$$(1)^2 = 2(0) - 1 + ce^0 \rightarrow 1 = -1 + c \rightarrow c = 2$$

$$y^2 = 2x - 1 + 2e^{-2x}.$$