MAC 2313 (Calculus III)
Test 2, Thursday October 26, 2006

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.

1. [12] Determine whether each of the following limit exits.

a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+2y^2}$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+2y^2} = \lim\limits_{x\to 0} \dfrac{0}{x^2} = 0 = \ell_1$ along x-axis

$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+2y^2} = \lim\limits_{x\to 0} \dfrac{x^2}{x^2+2x^2}$ along $y=x$

$= \lim\limits_{x\to 0} \dfrac{x^2}{3x^2}$

$= \frac{1}{3} = \ell_2$

Since $\ell_1 \neq \ell_2$, $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+2y^2}$ DNE

This problem was done in class

b) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2}$

Using spherical coordinats

$x = \rho\sin\varphi\cos\theta,\ y = \rho\sin\varphi\sin\theta,\ z = \rho\cos\varphi$

$\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2} = \lim\limits_{\rho\to 0} \dfrac{\rho^3\sin^2\varphi\cos\varphi\sin\theta\cos\theta}{\rho^2}$

$= \lim\limits_{\rho\to 0} \rho(\sin^2\varphi\cos\varphi \cdot \sin\theta\cos\theta)$

$= 0$, no matter how φ and θ are chosen.

Hence limit exists with value 0.

This problem was done in class

2. [12] Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0). \end{cases}$

Find $f_x(0,0)$, and $f_y(0,0)$. Is f differentiable at $(0,0)$?

$f_x(0,0) = \lim\limits_{x\to 0} \dfrac{f(x,0)-f(0,0)}{x}$

$= \lim\limits_{x\to 0} \dfrac{0}{x}$

$= 0$

$f_y(0,0) = \lim\limits_{y\to 0} \dfrac{f(0,y)-f(0,0)}{y}$

$= \lim\limits_{y\to 0} \dfrac{0}{y}$

$= 0$.

$\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)-f(0,0)-xf_x(0,0)-yf_y(0,0)}{\sqrt{x^2+y^2}}$

$= \lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{(x^2+y^2)\sqrt{x^2+y^2}}$.

Now $\lim\limits_{(x,y)\to(0,0)}$ along $y=x$ $= \lim\limits_{x\to 0} \dfrac{x^2}{2x^2\sqrt{2x^2}}$

$= \lim\limits_{x\to 0} \dfrac{1}{2\sqrt{2}|x|}$

$= \infty \neq 0$

So f is not diff-ble at $(0,0)$.

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3. [10] Consider the surface $xz + 2yz^2 - zy^2 = 1$. a) Find an equation for the tangent plane to the surface at the point $P(1,2,1)$. b) Find the parametric equations of the line normal to the surface at P.

a) Set $f(x,y,z) = xz + 2yz^2 - zy^2 - 1$; $f_x = z$, $f_y = 2z^2 - 2zy$, $f_z = x + 4yz - y^2$

$n = \nabla f(P)$ is a normal vector to tangent plane at P.

$= (1, -2, 5)$. Eqn of tangent plane is: $(x-1) - 2(y-2) + 5(z-1) = 0$

b) n is a direction of the normal line at P.

Parametric eqns are: $x = 1 + t$

$y = 2 - 2t$

$z = 1 + 5t$

4. [18] a) Find the velocity, the speed, and the acceleration, all of them at time $t = \pi/2$, of a particle moving along the curve $r(t) = e^t\vec{i} + e^t\sin(t)\vec{j} + e^t\cos(t)\vec{k}$. b) What is the curvature of that curve at each time t?

V = Velocity $= r'$. $V(t) = r'(t) = e^t\vec{i} + e^t(\cos t + \sin t)\vec{j} + e^t(-\sin t + \cos t)\vec{k}$

a = acceleration $= V'$. $a(t) = r''(t) = e^t\vec{i} + e^t(-\sin t + \cos t + \cos t + \sin t)\vec{j}$
$+ e^t(-\cos t - \sin t - \sin t + \cos t)\vec{k}$

$V(\frac{\pi}{2}) = e^{\frac{\pi}{2}}(\vec{i} + \vec{j} - \vec{k})$

speed $= \|V(\pi/2)\| = e^{\frac{\pi}{2}}\sqrt{1+1+1} = \sqrt{3}\,e^{\frac{\pi}{2}}$

$a(\frac{\pi}{2}) = e^{\frac{\pi}{2}}(\vec{i} - 2\vec{k})$

b) $$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} = \frac{\left\| \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ e^t & e^t(\cos t + \sin t) & e^t(-\sin t + \cos t) \\ e^t & 2e^t\cos t & -2e^t\sin t \end{vmatrix} \right\|}{\left(e^{2t} + e^{2t}(\cos t + \sin t)^2 + e^{2t}(-\sin t + \cos t)^2\right)^{3/2}}$$

$$= \frac{\|[-2e^{2t}(\sin t\cos t + \sin^2 t) - 2e^{2t}(-\sin t\cos t + \cos^2 t)]\vec{i} - [(-2e^{2t}\sin t - e^{2t}(-\sin t + \cos t)]\vec{j} + [2e^{2t}\cos t - e^{2t}(\cos t + \sin t)]\vec{k}\|}{e^{3t}(1 + 2(\cos^2 t + \sin^2 t))^{3/2}}$$

$$= \frac{e^{2t}\|-2\vec{i} + (\sin t + \cos t)\vec{j} + (\cos t - \sin t)\vec{k}\|}{e^{3t}(3)^{3/2}}$$

$$= \frac{e^{-t}\sqrt{4 + (\sin t + \cos t)^2 + (\cos t - \sin t)^2}}{3\sqrt{3}}$$

$$= \frac{e^{-t}\sqrt{6}}{3\sqrt{3}} = e^{-t}\frac{\sqrt{2}}{3}$$

5. [9+6] a) If $w^3 + x\cos(yz) = x$, use implicit differentiation to find w_x, w_y, and w_z. b) If $f(x, y, z) = y\cos(xz)$, find a local linear approximation L of the function f at $P(0, 1, 0)$, and use it to approximate $f(0.1, 1.1, 0.1)$.

a) $\frac{\partial}{\partial x}(w^3 + x\cos(yz)) = \frac{\partial}{\partial x}(x) \rightarrow 3w_x w^2 + \cos(yz) = 1 \rightarrow w_x = \frac{1 - \cos(yz)}{3w^2}$

$\frac{\partial}{\partial y}(w^3 + x\cos(yz)) = \frac{\partial}{\partial y}(x) \rightarrow 3w_y w^2 - xz\sin(yz) = 0 \rightarrow w_y = \frac{xz\sin(yz)}{3w^2}$

$\frac{\partial}{\partial z}(w^3 + x\cos(yz)) = \frac{\partial}{\partial z}(x) \rightarrow 3w_z w^2 - xy\sin(yz) = 0 \rightarrow w_z = \frac{xy\sin(yz)}{3w^2}$

b) $f_x = -yz\sin(xz)$, $f_y = \cos(xz)$, $f_z = -xy\sin(xz)$

$L(x, y, z) = f_x(P)x + f_y(P)(y-1) + f_z(P)z + f(P)$

$= 0 \cdot x + 1 \cdot (y-1) + 0 \cdot z + 1$

$= y - 1 + 1$

$= y$

$f(0.1, 1.1, 0.1) \approx L(0.1, 1.1, 0.1) = 1.1$

6. [15] a) If $x = uv$, $y = u^2 - v^2$, $z = u^2 + v^2$, and $w = \ln(1 + x + y + z)$, find w_u, w_v. Express your answers in terms of u and v. b) If $f(x, y, z) = xyz$, find the directional derivative of f at $P(1, 1, 1)$ in the direction from P to $Q(1, 2, -1)$.

a) $w_u = w_x \frac{\partial x}{\partial u} + w_y \frac{\partial y}{\partial u} + w_z \frac{\partial z}{\partial u}$

$= \frac{1}{1+x+y+z} \cdot v + \frac{1}{1+x+y+z} \cdot 2u + \frac{1}{1+x+y+z} \cdot 2u$

$= \frac{v + 4u}{1 + uv + 2u^2}$

$w_v = w_x \frac{\partial x}{\partial v} + w_y \frac{\partial y}{\partial v} + w_z \frac{\partial z}{\partial v}$

$= \frac{u}{1+x+y+z} + \frac{(-2v)}{1+x+y+z} + \frac{2v}{1+x+y+z}$

$= \frac{u}{1 + uv + 2u^2}$

b) $v = \overrightarrow{PQ} = (0, 1, -2)$. set $u = \frac{v}{\|v\|} = \frac{\vec{j} - 2\vec{k}}{\sqrt{5}}$. $\nabla f = \begin{pmatrix} yz \\ xz \\ xy \end{pmatrix}$

$D_v f = \nabla f(P) \cdot u$

$= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1/\sqrt{5} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

$= (1 - 2)/\sqrt{5}$

$= -\frac{1}{\sqrt{5}}$

7. [20] Let $f(x, y) = -x^3 + 6x + y^4 - 2y^2$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

Since f is diff-ble, C.P.s of f are given by $\nabla f = 0$.

$f_x = -3x^2 + 6$, $\quad f_y = 4y^3 - 4y$

$f_x = 0 \to 3x^2 = 6 \quad f_y = 0 \to y(y^2 - 1) = 0$

$\to x^2 = 2 \quad \to y = 0$ or $y = \pm 1$

$\to x = \pm\sqrt{2}$

C.P.s: $(-\sqrt{2}, -1)$, $(-\sqrt{2}, 0)$, $(-\sqrt{2}, 1)$, $(\sqrt{2}, -1)$, $(\sqrt{2}, 0)$, $(\sqrt{2}, 1)$.

$D = f_{xx} f_{yy} - f_{xy}^2 = -6x \cdot (12y^2 - 4)$

Point	$(-\sqrt{2}, -1)$	$(-\sqrt{2}, 0)$	$(-\sqrt{2}, 1)$	$(\sqrt{2}, -1)$	$(\sqrt{2}, 0)$	$(\sqrt{2}, 1)$
D / f_{xx}	$48\sqrt{2}$ / $6\sqrt{2}$	$-24\sqrt{2}$	$48\sqrt{2}$ / $6\sqrt{2}$	$-48\sqrt{2}$	$24\sqrt{2}$ / $-6\sqrt{2}$	$-48\sqrt{2}$
Classification	Local min	$D<0$ Saddle point	local min	$D<0$ Saddle point	local max.	$D<0$ Saddle point