MAC 2312 (Calculus II) — Answers
Test 2, Wednesday October 19, 2016

Name: **PID:**

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; guessing the correct answer won't earn you any credit. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Always do your best. Total=85 points. 3 pages

1. [10] Determine whether the improper integral converges or diverges. If it converges, state its limit, and if it diverges, state whether it diverges, due to oscillations, or to $+\infty$ or $-\infty$.

a) $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\sin x}{\sqrt{2\cos x - 1}}\,dx = \lim_{r\to\frac{\pi}{3}^-} \int_0^r \frac{\sin x\,dx}{\sqrt{2\cos x - 1}}$

$u = 2\cos x - 1$
$du = -2\sin x\,dx$

$\displaystyle = \lim_{r\to\frac{\pi}{3}^-} \int^{2\cos r - 1} -\frac{du}{2\sqrt{u}}$

$\displaystyle = -\lim_{r\to\frac{\pi}{3}^-} \sqrt{u}\,\Big]_1^{2\cos r - 1}$

$\displaystyle = -\left[\lim_{r\to\frac{\pi}{3}^-}(2\cos r - 1)^{1/2} - 1\right]$

$= -0 + 1$

$= 1$; so I_1 converges to 1

b) $\displaystyle\int_1^{+\infty} \frac{dx}{x^{\frac{10}{19}}} = \lim_{R\to+\infty} \int_1^R x^{-\frac{10}{19}}\,dx$

$\displaystyle = \lim_{R\to+\infty} \frac{x^{1-\frac{10}{19}}}{1-\frac{10}{19}}\Big]_1^R$

$\displaystyle = \lim_{R\to+\infty}\left(\frac{R^{9/19} - 1}{9/19}\right)$

$= +\infty$; so I_2 diverges to $+\infty$

2. [18] a) Write down the partial fractions decomposition of the rational function (Do not find the values of the numerical coefficients)

$$R(x) = \frac{2x^3 - 5x^2 + 7x - 11}{x^3\underbrace{(x^2+5x+6)}_{(x+2)(x+3)}(x^2+2)^2} = \frac{a}{x} + \frac{b}{x^2} + \frac{c}{x^3} + \frac{d}{x+2} + \frac{f}{x+3} + \frac{gx+h}{x^2+2} + \frac{kx+l}{(x^2+2)^2}$$

b) Evaluate the integrals

b1) $\displaystyle\int \frac{4x-5}{x(x^2+4)}\,dx = \int\left(\frac{a}{x} + \frac{bx+d}{x^2+4}\right)dx$

$\displaystyle = a\ln|x| + \frac{b}{2}\ln(x^2+4) + \frac{d}{2}\arctan(x/2) + C$

$\displaystyle\int\frac{dx}{x^2+4}$ $\quad x = 2\tan t$, $dx = 2\sec^2 t\,dt$

$\displaystyle = \int \frac{2\sec^2 t\,dt}{4\sec^2 t}$, as $\tan^2 t + 1 = \sec^2 t$

$\displaystyle = \int \frac{1}{2}\,dt$

$\displaystyle = \frac{t}{2} + C$

$\displaystyle = \frac{1}{2}\arctan(x/2)$

Now $\displaystyle\frac{4x-5}{x(x^2+4)} = \frac{a(x^2+4) + (bx+d)x}{x(x^2+4)}$

So $a(x^2+4) + x(bx+d) = 4x - 5$

For a, set $x = 0$: $4a = -5 \to a = -5/4$

For b & d, set $x = +2i$: $2i(2ib + d) = 8i - 5$

$-4b + 2id = 8i - 5$; so $-4b = -5$, $\underline{\underline{b = 5/4}}$

$2d = 8 \to \underline{\underline{d = 4}}$

b2) $\displaystyle\int \frac{x^2+4}{x+2}\,dx = \int\frac{(x^2-4+8)}{x+2}\,dx$

$\displaystyle = \int \frac{(x-2)(x+2)}{x+2} + \frac{8}{x+2}\,dx$

$\displaystyle = \int x - 2 + \frac{8}{x+2}\,dx$

$\displaystyle = \frac{(x-2)^2}{2} + 8\ln|x+2| + C$

3. [8] a) Approximate the integral $\int_1^3 \frac{1}{4x+3}\,dx$ using the midpoint rule with $n = 2$. b) Find an upper bound on the error in the approximation in a).

a) $x_0 = 1 \quad x_{01} = \frac{3}{2} \quad x_1 = 2 \quad x_{12} = \frac{5}{2} \quad x_2 = 3$

$$\int_1^3 \frac{1}{4x+3}\,dx \approx M_2 = \frac{2}{2}\left(\frac{1}{4(\frac{3}{2})+3} + \frac{1}{4(\frac{5}{2})+3}\right) = \frac{1}{9} + \frac{1}{13} = \frac{1}{9} + \frac{1}{13}$$

b) $f'(x) = -\frac{4}{(4x+3)^2}$, $f''(x) = \frac{32}{(4x+3)^3}$, where $f(x) = \frac{1}{4x+3}$. Now $0 < f''(x) \le f''(1)$

$$K_2 = f''(1) = \frac{32}{49}. \quad E_{M_2} = \left| \int_1^3 \frac{dx}{4x+3} - M_2 \right| \le \frac{2^3}{24(4)} \cdot \frac{32}{7(49)} = \frac{8}{3(49)7} = \frac{8}{21(49)}$$

4. [12] Decide whether each statement is true or false. No explanation needed.

a) If $\lim_{k\to\infty} k^2 u_k = 1$, then the series $\sum u_k$ converges. True, by the limit comparison test.

b) If the series $\sum u_k$ converges, then $\lim_{k\to\infty} u_k = 0$. True, by the divergence test.

c) If $\lim_{k\to\infty} \frac{|u_{k+1}|}{|u_k|} = 0.99$, then the series $\sum |u_k|$ converges. true, by the ratio test.

d) $\int_1^2 \frac{2x+1}{(x+1)(x+2)}\,dx$ is an improper integral. False, the integrand is continuous on $[1,2]$.

e) If $0 < a_k \le b_k$ for all $k \ge 1$, and $\sum a_k$ converges, then $\sum b_k$ converges too. False; pick $a_k = \frac{1}{k^2}$, $b_k = \frac{1}{k}$, $k \ge 1$

f) If $\lim_{k\to\infty} u_k = 0$, the the series $\sum u_k$ converges. False; pick $u_k = \frac{1}{k}$, $k \ge 1$.

5. [15] Determine whether the series converges or diverges. Be careful to explain your answers or else, get no credit.

a) $\sum_{k=1}^{\infty} \frac{k}{k+1}$; $\lim_{k\to\infty} \frac{k}{k+1} = 1 \ne 0$; So Series diverges by the divergence test

b) $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{2}{3}}}$; Series diverges as a p-series with $p < 1$

c) $\sum_{k=1}^{\infty} \frac{(-1)^k 3^{2k}}{2^{4k}}$; series converges as a geometric series with the ratio $r = -9/16$

d) Find the sum of the series in c). The initial term in the series is $-\frac{9}{16}$, therefore the series $\sum_{k=1}^{\infty} \frac{(-1)^k 3^{2k}}{2^{4k}} = \frac{-9/16}{1+\frac{9}{16}} = -\frac{9}{25}$

6. [10] Let $(u_n)_n$ be the sequence given by $u_n = \dfrac{2n+3}{7n-4}$, $n = 1, 2, \ldots$.

a) Write down the first four terms of the sequence $(u_n)_n$.

$$u_1 = \frac{5}{3}, \quad u_2 = \frac{7}{10}, \quad u_3 = \frac{9}{17}, \quad u_4 = \frac{11}{24}$$

b) Use the difference $u_{n+1} - u_n$ to show that the sequence $(u_n)_n$ is strictly decreasing.

$$u_{n+1} - u_n = \frac{2(n+1)+3}{7(n+1)-4} - \frac{2n+3}{7n-4} = \frac{(2n+5)(7n-4) - (2n+3)(7n+3)}{(7n+3)(7n-4)}$$
$$= \frac{\cancel{14n^2} - \cancel{8n} + 35n - 20 - (\cancel{14n^2} + \cancel{6n} + 21n + 9)}{(7n+3)(7n-4)}$$
$$= \frac{-29}{(7n+3)(7n-4)} < 0; \text{ so } (u_n)_n \text{ is strictly decreasing.}$$

c) Show that the sequence $(u_n)_n$ is bounded from below by zero. For each $n \geq 1$,

$2n+3 > 0$ and $7n-4 > 0$; so $u_n = \dfrac{2n+3}{7n-4} > 0$

d) Derive from b) and c) that $(u_n)_n$ converges, then find its limit.

Since $(u_n)_n$ is strictly decreasing and bounded from below, it follows that $(u_n)_n$ converges. $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty} \dfrac{2n+3}{7n-4} = \lim\limits_{n\to\infty} \dfrac{\cancel{n}(2+\frac{3}{n})}{\cancel{n}(7-\frac{4}{n})} = \dfrac{2}{7}$.

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7. [12] a) Use the ratio test to show that the series $\sum_{k=1}^{\infty} \dfrac{(-1)^k k!}{1.3.5.7.....(2k-1)}$ converges absolutely.

$$\rho = \lim_{k\to\infty} \frac{(k+1)!}{1.3.5\ldots(2k-1)(2k+1)} \cdot \frac{1.3.5\ldots(2k-1)}{k!} = \lim_{k\to\infty} \frac{\cancel{k!}(k+1)\cdot \cancel{1.3.5\ldots(2k-1)}}{\cancel{k!}\,\cancel{1.3.5\ldots(2k-1)}(2k+1)}$$
$$= \lim_{k\to\infty} \frac{k+1}{2k+1}$$
$$= \lim_{k\to\infty} \frac{\cancel{k}(1+\frac{1}{k})}{\cancel{k}(2+\frac{1}{k})} = \frac{1}{2} < 1; \text{ so}$$

Series converges absolutely.

b) Use the limit comparison test to decide whether the series $\sum_{k=1}^{\infty} \dfrac{\sqrt{k}}{k^3+2}$ converges or diverges.

$$\frac{\sqrt{k}}{k^3+2} = \frac{\sqrt{k}}{k^3(1+\frac{2}{k^3})} = \frac{k^{-5/2}}{1+\frac{2}{k^3}}; \quad \text{set } u_k = \frac{\sqrt{k}}{k^3+2}, \quad v_k = k^{-5/2} = \frac{1}{k^{5/2}}$$

$\lim\limits_{k\to\infty} \dfrac{u_k}{v_k} = \lim\limits_{k\to\infty} \dfrac{1}{1+\frac{2}{k^3}} = 1$; the series $\sum_{k=1}^{\infty} v_k$ converges; p-series, $p = \frac{5}{2} > 1$,

So $\sum u_k$ converges by the limit comparison test.