MAC 2313 (Calculus III) — Answers

Test 2, Wednesday October 19, 2016

Name: **PID:**

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve any of the points assigned to any question. *You will not get any credit if you do not show the steps to your answers.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. 3 pages. Total=85 points. Always do your best.**

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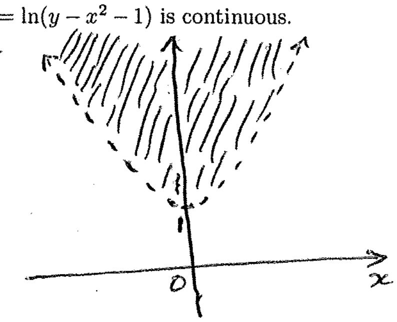
1 . [15] a) Describe and sketch the largest region where the function h defined by $h(x,y) = \ln(y - x^2 - 1)$ is continuous.

h is continuous at all points (x,y) satisfying $y - x^2 - 1 > 0$ or $y > x^2+1$. Sketch $y = x^2+1$; points (x,y) for which $y > x^2+1$ are points above the parabola



b) Describe in words, the domain of the function f given by $f(x,y,z) = \sqrt{x^2+4y^2-z^2}$

$D_f = \{(x,y,z) \text{ in } \mathbb{R}^3;\ x^2+4y^2-z^2 \geq 0 \text{ or } z^2 \leq x^2+4y^2\}$

= all points in 3-space on or below the cone $z = \sqrt{x^2+4y^2}$, and all points on or above the cone $z = -\sqrt{x^2+4y^2}$

c) Find an equation for the level surface of the function g defined by $g(x,y,z) = \int_x^{yz} \frac{t}{t^2+1}\,dt$ that passes through the point $P(1, \sqrt{3}, 1)$.

$g(x,y,z) = \frac{1}{2}\ln(1+t^2)\Big]_x^{yz} = \frac{1}{2}\left(\ln(1+y^2z^2) - \ln(1+x^2)\right)$

level surface equation: $\frac{1}{2}\ln\left(\frac{1+y^2z^2}{1+x^2}\right) = g(P) = \frac{1}{2}\ln\left(\frac{1+3}{1+1}\right) = \frac{1}{2}\ln 2$

So level surface simplifies to $\dfrac{1+y^2z^2}{1+x^2} = 2.$

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2. [15] a) Write down the definition of "f is differentiable at (x_0, y_0)". b) Use the definition in a) to show that the function f given by $f(x,y) = 2xy - y^2$ is differentiable at $(1,-1)$.

a) f is differentiable at (x_0,y_0) if f as well as f_x and f_y are defined at (x_0,y_0) and $\displaystyle\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}} = 0$

b) $f_x(x,y) = 2y$, $f_y(x,y) = 2x - 2y$

f, f_x and f_y are polynomials; so they are all defined at $(1,-1)$.

Now $f(1,-1) = -2-1 = -3$, $f_x(1,-1) = -2$, $f_y(1,-1) = 2+2 = 4$

$\displaystyle\lim_{(h,k)\to(0,0)} \frac{f(1+h,-1+k)+3+2h-4k}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{2(1+h)(-1+k) - (-1+k)^2 + 3 + 2h - 4k}{\sqrt{h^2+k^2}}$

$\displaystyle = \lim_{(h,k)\to(0,0)} \frac{-\cancel{2}+\cancel{2k}-\cancel{2h}+2hk-(\cancel{1}-\cancel{2k}+k^2)+\cancel{3}+\cancel{2h}-\cancel{4k}}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{2hk-k^2}{\sqrt{h^2+k^2}}$; set $h = r\cos\theta$, $k = r\sin\theta$, $r > 0$, $0 \leq \theta \leq 2\pi$

$\displaystyle = \lim_{r\to 0^+} \frac{2r^2\cos\theta\sin\theta - r^2\sin^2\theta}{r} = \lim_{r\to 0^+} r(\cos\theta - \sin\theta)\sin\theta$

$= 0$ for all θ; so f is differentiable at $(1,-1)$

3. [10] Decide whether the statement is true or false. No explanation is needed.

a) If $z(t) = f(x(t), y(t))$, then $\frac{dz}{dt}(t) = f_x(\frac{dx}{dt}, y) + f_y(x, \frac{dy}{dt})$. False, See Theorem 13.5.1

b) If $f_x(1,2)$ and $f_y(1,2)$ both exist, then f is continuous at $(1,2)$. False, pick $f(x,y) = \begin{cases} \frac{(x-1)(y-2)}{(x-1)^2+(y-2)^2} & \text{if } (x,y) \neq (1,2) \\ 0 & \text{if } (x,y) = (1,2) \end{cases}$

c) If f is differentiable at $(7,2,-9)$, then f is continuous at $(7,2,-9)$. True, Theorem 13.4.3

d) If $\lim_{(x,y)\to(-1,1)} f(x,y) = 3$, then $f(x,y) \to 3$ as (x,y) approaches $(-1,1)$ along the line $y = 1$ and $f(x,y) \to 3$ as (x,y) approaches $(-1,1)$ along the parabola $y = 1 + (x+1)^2$. True, Theorem 13.2.2

e) If $f = f(x,y,z)$ is differentiable at the point $B(-1,4,-5)$, then the directional derivative of f at B in the direction of the vector $\vec{r} = \frac{1}{2}(\vec{i} - \sqrt{2}\vec{j} + \vec{k})$ is given by $D_{\vec{r}} f(B) = \nabla f(B) \cdot \vec{r}$. True, by Theorem 13.6.3

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4. [7] Evaluate each limit. If a limit does not exist, explain why.

a) $\lim_{(x,y,z)\to(-1,1,2)} \frac{xyz}{x^2+y^2+z^2} = \frac{-1(1)(2)}{1+1+4} = -\frac{2}{6} = -\frac{1}{3}$

b) $\lim_{(u,v)\to(1,-1)} \frac{u^3+v^3}{u^2-v^2} = \lim_{(u,v)\to(1,-1)} \frac{\cancel{(u+v)}(u^2-uv+v^2)}{\cancel{(u+v)}(u-v)} = \lim_{(u,v)\to(1,-1)} \frac{u^2-uv+v^2}{u-v} = \frac{1+1+1}{1+1} = \frac{3}{2}$

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5. [20] a) Find an equation for the tangent plane and parametric equations for the normal line to the surface $3\sqrt{x^2+y^2} - 2\sqrt{y^2+z^2} = 1$ at the point $(1,0,1)$.

Set $F(x,y,z) = 3\sqrt{x^2+y^2} - 2\sqrt{y^2+z^2} - 1$; Equation of surface is then $F(x,y,z) = 0$

$\nabla F(x,y,z) = \left\langle \frac{3x}{\sqrt{x^2+y^2}}, \frac{3y}{\sqrt{x^2+y^2}} - \frac{2y}{\sqrt{y^2+z^2}}, \frac{-2z}{\sqrt{y^2+z^2}} \right\rangle$, $\nabla F(1,0,1) = \langle 3, 0, -2 \rangle$
= a normal to tangent plane
= a parallel vector to normal line

Equation of tangent plane: $3(x-1) - 2(z-1) = 0$

parametric equations of normal line

$x = 1 + 3t$, $y = 0$, $z = 1 - 2t$, $-\infty < t < \infty$

b) Set $h(x,y,z) = xy\cos(yz)$. Find the local linear approximation of h about the point $P(1, 1/2, \pi)$, then, use it to approximate $h(1.01, 0.498, (0.99)\pi)$.

$\nabla h(x,y,z) = \langle y\cos(yz), x\cos(yz) - xyz\sin(yz), -xy^2\sin(yz) \rangle$

$\nabla h(P) = \langle 0, -\frac{\pi}{2}, -\frac{1}{4} \rangle$, as $\cos(\frac{\pi}{2}) = 0$, $\sin(\frac{\pi}{2}) = 1$.

$L(x,y,z) = -\frac{\pi}{2}(y - \frac{1}{2}) - \frac{1}{4}(z - \pi)$ = local linear approximation of h about P

$h(1.01, 0.498, (0.99)\pi) \approx L(1.01, 0.498, (0.99)\pi) = -\frac{\pi}{2}(-0.002) - \frac{1}{4}(-0.01)\pi$

$\approx \pi\left((0.001) + \frac{0.01}{4}\right)$

c) Find a unit vector in the direction in which the function h in b) increases most rapidly at the point P, and find the rate of change of h at P in that direction.

$\vec{u} = \frac{\nabla h(P)}{\|\nabla h(P)\|} = \frac{\langle 0, -\frac{\pi}{2}, -\frac{1}{4} \rangle}{\sqrt{\frac{\pi^2}{4} + \frac{1}{16}}}$

rate of change at P in the direction of $\vec{u}$ = $D_{\vec{u}} h(P) = \nabla h(P) \cdot \vec{u}$
$= \|\nabla h(P)\|$
$= \sqrt{\frac{\pi^2}{4} + \frac{1}{16}}$

6. [12] Let $f(x,y) = x^2 + x^2y - y^2 - 4y + 1$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

$f_x(x,y) = 2x + 2xy, \quad f_y(x,y) = x^2 - 2y - 4$

$\nabla f(x,y) = \vec{0} \longrightarrow x + xy = 0$ or $x(1+y) = 0$; so $x = 0$ or $y = -1$

& $x^2 - 2y - 4 = 0$ or $x^2 = 2y + 4$

Now if $x = 0$, then $-2y = 4$ or $y = -2$; if $y = -1$, then $x^2 = -2 + 4 = 2$

So $x = \pm\sqrt{2}$.

C.Ps: $(0,-2)$, $(-\sqrt{2},-1)$, $(\sqrt{2},-1)$

$f_{xx}(x,y) = 2 + 2y, \quad f_{xy}(x,y) = 2x, \quad f_{yy}(x,y) = -2$

$\Delta(x,y) = 4x^2 - (2+2y)(-2) = 4x^2 + 4(1+y)$

(x_0,y_0)	$\Delta(x_0,y_0)$	$f_{xx}(x_0,y_0)$	Classification
$(0,-2)$	-4	-2	local maximum
$(-\sqrt{2},-1)$	8		Saddle point
$(\sqrt{2},-1)$	8		Saddle point

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7. [6] Use implicit partial differentiation to find the partial derivatives $\frac{\partial y}{\partial x}$ and $\frac{\partial y}{\partial z}$ if $yz + x\sin(xy) = 1$.

$$\frac{\partial}{\partial x}\left(yz + x\sin(xy)\right) = \frac{\partial}{\partial x}(1) = 0$$

$$z\frac{\partial y}{\partial x} + \sin(xy) + x\left(y + x\frac{\partial y}{\partial x}\right)\cos(xy) = 0$$

$$\left(z + x^2\cos(xy)\right)\frac{\partial y}{\partial x} = -\sin(xy) - xy\cos(xy)$$

$$\frac{\partial y}{\partial x} = \frac{-\sin(xy) - xy\cos(xy)}{z + x^2\cos(xy)}$$

$$\frac{\partial}{\partial z}\left(yz + x\sin(xy)\right) = \frac{\partial}{\partial z}(1) = 0$$

$$z\frac{\partial y}{\partial z} + y + x^2\frac{\partial y}{\partial z}\cos(xy) = 0$$

$$\left(z + x^2\cos(xy)\right)\frac{\partial y}{\partial z} = -y$$

$$\frac{\partial y}{\partial z} = -\frac{y}{z + x^2\cos(xy)}.$$