MAC 2313 (Calculus III) – Answers
Test 2, Friday May 26, 2017

Name: **PID:**

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. 2 pages. Total=60 points. Always do your best.

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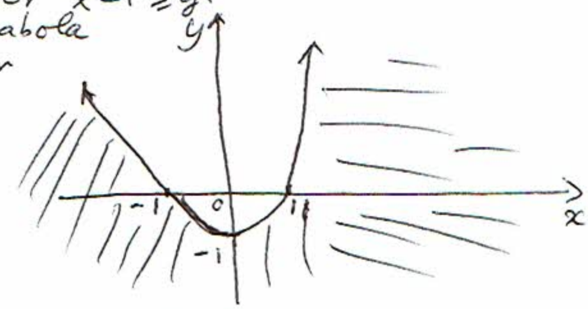
1 . [15] a) Describe and sketch the largest region where the function h defined by $h(x,y) = \sqrt{x^2 - y - 1}$ is continuous.

h is continuous for $x^2 - y - 1 \geq 0$ or $x^2 - 1 \geq y$.
Now, sketch $y = x^2 - 1$, which a parabola
$x^2 - 1 \geq y$ is the region below and on the parabola
h is continuous on and below the parabola $y = x^2 - 1$.



b) If $f(x,y,z) = 2xy - yz + 3zx$, find the directional derivative of f at the point $A(2,1,1)$ in the direction of the vector $\overrightarrow{w} = \overrightarrow{i} - 2\overrightarrow{j} + 3\overrightarrow{k}$.

$D_{\overrightarrow{w}} f(A) = \nabla f(A) \cdot \frac{\overrightarrow{w}}{\|\overrightarrow{w}\|}$. Now $\nabla f(x,y,z) = \langle 2y + 3z, 2x - z, -y + 3x \rangle$

$\nabla f(A) = \langle 5, 3, 5 \rangle$, $\|\overrightarrow{w}\| = \sqrt{1+4+9} = \sqrt{14}$; $\nabla f(A) \cdot \overrightarrow{w} = 5(1) + 3(-2) + 5(3) = 14$

Hence $D_{\overrightarrow{w}} f(A) = \frac{14}{\sqrt{14}} = \sqrt{14}$.

c) Find a unit vector in the direction in which the function f in b) decreases most rapidly at the point A, and find the rate of change of f at A in that direction.

The function f decreases most rapidly in the direction of $-\nabla f(A)$
So, the required unit vector is $\overrightarrow{u} = -\frac{\nabla f(A)}{\|\nabla f(A)\|} = \frac{1}{\sqrt{59}} \langle -5, -3, -5 \rangle$
and the rate of change of f in the direction of $-\nabla f(A)$ is
$-\|\nabla f(A)\| = -\sqrt{59}$

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2. [15] a) Write down the definition of "f is differentiable at (x_0, y_0)". b) Use the definition in a) to show that the function f given by $f(x,y) = 4xy - 3$ is differentiable at $(0,1)$.

a) $\displaystyle\lim_{(h,k)\to(0,0)} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - h f_x(x_0,y_0) - k f_y(x_0,y_0)}{\sqrt{h^2+k^2}} = 0$

b) $f(0,1) = -3$, $f_x(x,y) = 4y$, $f_y(x,y) = 4x$, $f_x(0,1) = 4$, $f_y(0,1) = 0$

$\displaystyle\lim_{(h,k)\to(0,0)} \frac{4h(1+k) - 3 + 3 - 4h}{\sqrt{h^2+k^2}} = \lim_{(h,k)\to(0,0)} \frac{4hk}{\sqrt{h^2+k^2}}$, $h = r\cos\theta$, $k = r\sin\theta$, $r > 0$, $0 \leq \theta \leq 2\pi$

$= \displaystyle\lim_{r\to 0^+} \frac{4r^2\cos\theta\sin\theta}{r}$

$= 4(\displaystyle\lim_{r\to 0^+} r)\cos\theta\sin\theta$

$= 0$, for all θ in $[0, 2\pi]$

Hence f is differentiable at $(0,1)$.

3. [10] a) Find an equation for the tangent plane and parametric equations for the normal line to the surface $2x^2-3y^2+4z^2=3$ at the point $B(1,-1,1)$. b) Find an equation for the level curve of the function $g(x,y)=\sqrt{x^2+y^2}$ that passes through the point $(-4,3)$. Describe that level curve.

Set $f(x,y,z) = 2x^2-3y^2+4z^2-3$. $\nabla f(x,y,z) = \langle 4x, -6y, 8z \rangle$

$\nabla f(B) = \langle 4, 6, 8 \rangle$ = a normal to tangent plane = a vector parallel to normal line

Equation of tangent plane: $4(x-1)+6(y+1)+8(z-1)=0$ or $2(x-1)+3(y+1)+4(z-1)=0$

Parametric equations of normal line:
$x = 1+2t$
$y = -1+3t$
$z = 1+4t$

b) $g(-4,3) = \sqrt{16+9} = 5$, level curve: $\sqrt{x^2+y^2} = 5$ or $x^2+y^2=25$

Level curve is the circle centered at $(0,0)$ with radius 5.

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4. [7] Evaluate each limit. If a limit does not exist, explain why.

a) $\displaystyle\lim_{(x,y,z)\to(-1,1,2)} \frac{xy-5yz}{\sqrt{x^2-2y^2+z^2}} = \frac{-1(1)-5(1)(2)}{\sqrt{1-2+4}} = \frac{-11}{\sqrt{3}}$

b) $\displaystyle\lim_{(u,v)\to(-1,1)} \frac{\left(u^2-v^2\right)^2}{u^2+2uv+v^2}$

$\displaystyle = \lim_{(u,v)\to(-1,1)} \frac{(u-v)^2(u+v)^2}{(u+v)^2}$

$\displaystyle = \lim_{(u,v)\to(-1,1)} (u-v)^2$

$= (-1-1)^2 = 4$

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5. [13] Let $f(x,y)=x^3-3xy+6y^2-5$. Find all the critical points of f and classify each of them as a local maximum, a local minimum, or a saddle point.

$f_x(x,y) = 3x^2-3y$
$f_y(x,y) = -3x+12y$

$f_x(x,y)=0 \to y=x^2$
$f_y(x,y)=0 \to x=4y$
$\to y = (4y)^2 = 16y^2$

$y-16y^2=0 \quad y(1-16y)=0 \to y=0$ or $y=\frac{1}{16}$

$y=0 \to x=0$
$y=\frac{1}{16} \to x=1/4=4^{-1}$
C.Ps: $(0,0)$, $(4^{-1}, \frac{1}{16})$

$f_{xx}(x,y)=6x$, $f_{yy}(x,y)=12$
$f_{xy}(x,y)=-3$

$\Delta(x,y) = f_{xy}(x,y)^2 - f_{xx}(x,y)f_{yy}(x,y)$
$= 9-72x$

C.P	$(0,0)$	$(\frac{1}{4},\frac{1}{16})$
$\Delta(x_0,y_0)$	9	-9
$f_{xx}(x_0,y_0)$		$3/2$
Class	saddle point	local minimum