MAP 2302 (Differential Equations) — Answers
TEST 2, Friday April 1, 2016

Name: PID:

Remember that no documents or calculators are allowed during the test. You must show all your work to deserve the full credit assigned to any question. 4 pages.

1.[10] a) Show that the two functions $\cos x$ and $x\cos x$ are linearly independent on the interval $(0,1)$.

$$W(\cos x, x\cos x) = \begin{vmatrix} \cos x & x\cos x \\ -\sin x & \cos x - x\sin x \end{vmatrix}$$
$$= \cos^2 x - x\sin x\cos x + x\sin x\cos x$$
$$= \cos^2 x \neq 0 \text{ for all } x \text{ in } (0,1); \text{ so } \cos x \text{ and}$$

$x\cos x$ are linearly independent on $(0,1)$.

b) Given that 8,-4,-4,-3i,3i,0,5-2i,5+2i,5-2i,5+2i,6-7i,6+7i are the roots of the auxiliary equation corresponding to some 12th-order homogeneous linear differential equation with constant coefficients, write down the general solution of the differential equation.

$$y = c_1 e^{8x} + (c_2 + c_3 x)e^{-4x} + c_4 \sin(3x) + c_5 \cos(3x) + c_6$$
$$+ \left((c_7 + c_8 x)\sin(2x) + (c_9 + c_{10} x)\cos(2x)\right)e^{5x} + e^{6x}\left(c_{11}\cos(7x) + c_{12}\sin(7x)\right)$$

$c_1, c_2, \ldots, c_{12}$ = arbitrary constants

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2. [10] Solve the differential equation: $y'' - 2y' - y = 0$.

Aux. equation: $m^2 - 2m - 1 = 0$
$$(m-1)^2 - 2 = 0$$
$$(m - 1 - \sqrt{2})(m - 1 + \sqrt{2}) = 0$$
$$m_1 = 1 + \sqrt{2}, \quad m_2 = 1 - \sqrt{2}$$
$$y = c_1 e^{(1+\sqrt{2})x} + c_2 e^{(1-\sqrt{2})x}, \quad c_1, c_2 = \text{Constants}$$

3. [20] Solve the Cauchy-Euler equation: $x^2y'' - xy' + 5y = 15\ln x, \quad x > 0.$

Set $x = e^t$; $t = \ln x$, $\frac{dt}{dx} = \frac{1}{x}$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}; \text{ So } \frac{dy}{dt} = x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(x^{-1}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + x^{-1}\frac{d^2y}{dt^2}\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}; \text{ so } \frac{d^2y}{dt^2} - \frac{dy}{dt} = x^2\frac{d^2y}{dx^2}.$$

D.E becomes:

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + 5y = 15t$$

or
$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 15t$$

Homogeneous eqn: $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 5y = 0$

Aux. eqn: $m^2 - 2m + 5 = 0$, $(m-1)^2 + 4 = 0$

$(m-1-2i)(m-1+2i) = 0$

$m_1 = 1+2i$, $m_2 = 1-2i$; $y_c = (c_1 \sin(2t) + c_2\cos(2t))e^t$

$= (c_1 \sin(2\ln x) + c_2 \cos(2\ln x))x$

Method of undetermined coefficients:

$c_1, c_2 =$ constants.

$S_{15t} = \{1, t\}$

Seek a particular soln

$y_p = A + Bt$, $A, B =$ constants

$y_p' = B$, $y_p'' = 0$

$y_p'' - 2y_p' + 5y_p = -2B + 5A + 5Bt = 15t$

$5A - 2B = 0 \longrightarrow 5A = 2B = 6 \rightarrow A = 6/5$

$5B = 15 \rightarrow B = 3$

$$y_p = 3t + \frac{6}{5} = 3\ln x + \frac{6}{5}$$

$$y = y_c + y_p = c_1(\sin(2\ln x) + c_2\cos(2\ln x))x + 3\ln x + \frac{6}{5}$$

4. [20] Use the variation of parameters method to solve the differential equation:
$(\cos^2 x)y'' + 2(\sin x \cos x)y' + (\sin^2 x + 1)y = 2\cos^3 x$, given that $y_1(x) = \cos x$ and $y_2(x) = x\cos x$ are linearly independent solutions of the corresponding homogeneous equation.

$$y_c = c_1 \cos x + c_2 x \cos x \qquad c_1, c_2 = \text{constants}$$

Seek a particular solution $y_p = v_1(x)\cos x + v_2(x)\, x\cos x$ with

(1) $\quad v_1' \cos x + v_2' x\cos x = 0$

(2) $\quad -v_1' \sin x + v_2'(\cos x - x\sin x) = \dfrac{2\cos^3 x}{\cos^2 x} = 2\cos x$

$(\sin x)\cdot(1) + (\cos x)\cdot(2)$:

$$v_2' x \cancel{\sin x \cos x} + v_2'(\cos^2 x - \cancel{x\sin x \cos x}) = 2\cos^2 x$$

$$v_2' \cos^2 x = 2\cos^2 x \rightarrow v_2' = 2 \rightarrow v_2(x) = \int 2\,dx = 2x$$

(1) yields: $v_1' = -x v_2' = -2x \rightarrow v_1(x) = \int -2x\,dx = -x^2$

Hence

$$y_p(x) = -x^2 \cos x + 2x^2 \cos x = x^2 \cos x$$

$$y = y_c + y_p = c_1 \cos x + c_2 x\cos x + x^2 \cos x$$

5. [25] a) Use the definition of the Laplace transform to find the Laplace transform of $f(t) = \sin(2t)$

$$\mathcal{L}(f)(s) = \int_0^{\infty} e^{-st}\sin(2t)\,dt = \lim_{R\to\infty}\int_0^R e^{-st}\sin(2t)\,dt$$

$$= \lim_{R\to\infty}\left(-\frac{e^{-st}}{s}\sin(2t)\Big]_0^R + \frac{1}{s}\int_0^R e^{-st}(2\cos(2t))\,dt\right)$$

$$= \frac{2}{s}\lim_{R\to\infty}\int_0^R e^{-st}\cos(2t)\,dt, \text{ for all } s>0, \text{ as } \lim_{R\to\infty}\frac{e^{-sR}}{s}\sin(2R)=0$$

$$= \frac{2}{s}\lim_{R\to\infty}\left(-\frac{e^{-st}}{s}\cos(2t)\Big]_0^R + \frac{1}{s}\int_0^R e^{-st}(-2\sin(2t))\,dt\right)$$

$$= \frac{2}{s}\left(\lim_{R\to\infty} -\frac{e^{-Rs}}{s}\cos(2R) + \frac{1}{s} - \frac{2}{s}\mathcal{L}(f)(s)\right)$$

$$= \frac{2}{s^2} - \frac{4}{s^2}\mathcal{L}(f)(s);\ \text{So } \left(1+\frac{4}{s^2}\right)\mathcal{L}(f)(s) = \frac{2}{s^2}$$

Hence $\mathcal{L}(\sin(2t))(s) = \dfrac{2}{s^2+4}, \quad s>0.$

b) Use the properties of Laplace transform to find the following Laplace transforms (Show all your work):

i) $\mathcal{L}(e^{-3t}\cos^2(3t))(s) = \mathcal{L}\left(e^{-3t}\left(\frac{\cos(6t)+1}{2}\right)\right)(s)$

$$= \frac{1}{2}\mathcal{L}(e^{-3t}\cos(6t))(s) + \frac{1}{2}\mathcal{L}(e^{-3t})(s)$$

$$= \frac{1}{2}\mathcal{L}(\cos(6t))(s+3) + \frac{1}{2(s+3)}, \text{ by the translation property}$$

$$= \frac{s+3}{2((s+3)^2+36)} + \frac{1}{2(s+3)}$$

ii) $\mathcal{L}(te^{2t}\cos t)(s) = -\frac{d}{ds}\mathcal{L}(e^{2t}\cos t)(s) = -\frac{d}{ds}\left(\frac{s-2}{(s-2)^2+1}\right)$, by

$$= -\left(\frac{(s-2)^2+1 - 2(s-2)^2}{((s-2)^2+1)^2}\right) = \frac{(s-2)^2-1}{((s-2)^2+1)^2}$$

iii) $\mathcal{L}(4t^3e^{5t} - 7e^t\sin(5t))(s) = 4\mathcal{L}(t^3)(s-5) - 7\mathcal{L}(\sin(5t))(s-1)$

$$= \frac{4(6)}{(s-5)^4} - \frac{7(5)}{(s-1)^2+25}$$

$$= \frac{24}{(s-5)^4} - \frac{35}{(s-1)^2+25}$$