MAS 3105 (Linear Algebra) — key
Test 2, Friday June 05, 2015

Name: PID:

Remember that you won't get any credit if you do not show the steps to your answers. Total=115 points.

1. [20] Find a basis for the null space and a basis for the column space of the matrix $A = \begin{pmatrix} -1 & 1 & 2 & 3 \\ 2 & 2 & 3 & -5 \\ 1 & 7 & 12 & -1 \end{pmatrix}$.

To find $N(A)$, find the RREF U of A, and solve $Ux = 0_{\mathbb{R}^3}$; the lead variables tell which column of A to use for a basis for the column space $R(A)$ of A. Let's find U.

$$A \xrightarrow[r_1+r_3]{2r_1+r_2} \begin{pmatrix} -1 & 1 & 2 & 3 \\ 0 & 4 & 7 & 1 \\ 0 & 8 & 14 & 2 \end{pmatrix} \xrightarrow{-2r_2+r_3} \begin{pmatrix} -1 & 1 & 2 & 3 \\ 0 & 4 & 7 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\substack{r_2/4 \\ -r_1}]{-r_2/4+r_1} \underbrace{\begin{pmatrix} +1 & 0 & -\frac{1}{4} & -\frac{11}{4} \\ 0 & 1 & \frac{7}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{U}$$

lead variables are x_1 & x_2; so $R(A)$ has $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix} \right\}$ as a basis.

$$Ux = 0_{\mathbb{R}^3} \longrightarrow \quad x_2 + \frac{7}{4}x_3 + \frac{x_4}{4} = 0 \longrightarrow x_2 = -\frac{7}{4}x_3 - \frac{x_4}{4}$$

$$x_1 - \frac{x_3}{4} - \frac{11}{4}x_4 = 0 \longrightarrow x_1 = \frac{x_3}{4} + \frac{11}{4}x_4$$

$$N(A) = \text{Span}\left((1, -7, 4, 0)^T, (11, -1, 0, 4)^T \right)$$

So $\left\{ \begin{pmatrix} 1 \\ -7 \\ 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ -1 \\ 0 \\ 4 \end{pmatrix} \right\}$ is a basis for $N(A)$.

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2. [20] State whether each of the following statement is true or false. No explanations needed.

a) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_8$ are linearly independent , then they span $\mathbb{R}^8$. True, by Theorem 3.4.3

b) If A is a 9×15 matrix , then A and A^T have the same rank. True, by Theorem 3.6.6

c) If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_9$ are linearly independent vectors in a vector space E, then $\mathbf{x}_1$, $\mathbf{x}_3$, $\mathbf{x}_5$, and $\mathbf{x}_8$ are linearly independent. True, Homework pb 15 in 3.3

d) If U is the reduced row echelon form of a matrix A, then A and U have the same column space. False, see Note

e) If $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5$ span a subspace of $\mathbb{R}^5$, then they are linearly independent. False as dim(subspace) may be less than 5

f) If A is a 12×10 matrix , then A and A^T have the same nullity. False, by rank-nullity Theorem

g) If L is a linear operator on $\mathbb{R}^n$ and $\mathbf{x} \in \ker(L)$, then $L(\mathbf{x} + \mathbf{z}) = L(\mathbf{z})$, for each $\mathbf{z} \in \mathbb{R}^n$. True as $L(x) = 0_{\mathbb{R}^n}$

h) If U is the reduced row echelon form of a 10×14 matrix A, then U and A have the same row space. True, by Th. 3.6.1

i) If L is a linear operator on $\mathbb{R}^6$ with $R(L) = \mathbb{R}^6$, and A is the standard matrix for L, then A is nonsingular. True as $r_A = 6$

j) If A and C are $n \times n$ matrices such that A is nonsingular and A is similar to C, then C is nonsingular and A^{-1} is similar to C^{-1}. True

$A = P^{-1}CP$ for some nonsingular $n \times n$ matrix P

$C = PAP^{-1}$, so C is nonsingular as a product of nonsingular matrices

$C^{-1} = PA^{-1}P^{-1} \Longleftrightarrow A^{-1} = P^{-1}C^{-1}P$; so A^{-1} & C^{-1} are similar

3. [10] Let $\mathcal{M}_2 = \left\{ A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}; a,\ b,\ c,\ d \in \mathbb{R} \right\}$ be the space of 2×2 matrices. Define on $\mathcal{M}_2$ a mapping L by $L(A) = A - A^T$. a) Show that L is linear. b) Find a basis for $\ker(L)$ and a basis for $\mathrm{R}(L)$.

a) Let $\alpha \in \mathbb{R}$, A, B in $\mathcal{M}_2$.

$L(A+\alpha B) = A + \alpha B - (A+\alpha B)^T = A + \alpha B - A^T - \alpha B^T = A - A^T + \alpha(B - B^T) =$
$L(A) + \alpha L(B)$; hence L is linear.

b) $A \in \ker(L)$ iff $L(A) = 0_{\mathcal{M}_2} \Leftrightarrow A = A^T$; so $\ker(L) = \{A \in \mathcal{M}_2;\ A$ is symmetric$\}$
$\ker(L) = \{A = \begin{pmatrix} a & b \\ b & d \end{pmatrix};\ a, b, c, d \in \mathbb{R}\}$. $A \in \ker(L)$ iff $A = aE_{11} + b(E_{12} + E_{21}) + dE_{22}$

Hence $\{E_{11},\ E_{12} + E_{21},\ E_{22}\}$ is a basis for $\ker(L)$.

$B \in R(L)$ iff $B = L(A)$ for some $A \in \mathcal{M}_2 \Leftrightarrow B = \begin{pmatrix} 0 & c-b \\ b-c & 0 \end{pmatrix}$ if $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Hence $\{E_{21} - E_{12}\}$ is a basis for $R(L)$. $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

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4. [30] Let $\mathbf{v}_1 = (-1,-2,1)^T$, $\mathbf{v}_2 = (1,3,2)^T$, $\mathbf{v}_3 = (1,1,2)^T$, and $\mathbf{w}_1 = (-1,-3,1)^T$, $\mathbf{w}_2 = (2,3,1)^T$ and $\mathbf{w}_3 = (1,1,3)^T$ be vectors in $\mathbb{R}^3$. Let L be the linear operator defined on $\mathbb{R}^3$ by

$L(x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + x_3\mathbf{w}_3) = (x_1 - x_2 + x_3)\mathbf{w}_1 + (x_2 - x_3 + x_1)\mathbf{w}_2 + (x_3 - x_1 + x_2)\mathbf{w}_3.$

a) Find the matrix representation M of L relative to the ordered basis $B = [\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3]$. b) Find the transition matrix T from the ordered basis B to the ordered basis $D = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$. c) Write down the matrix P of L with respect to the ordered basis D in terms of M, but do not attempt to find the entries of P. d) If $\mathbf{z} = 2\mathbf{w}_1 - \mathbf{w}_2 + 3\mathbf{w}_3$, find the coordinates of $L(\mathbf{z})$ in the ordered basis D.

a) $L(w_1) = w_1 + w_2 - w_3$, as $w_1 = 1w_1 + 0w_2 + 0w_3$
$L(w_2) = -w_1 + w_2 + w_3$, $\quad w_2 = 0w_1 + 1w_2 + 0w_3$
$L(w_3) = w_1 - w_2 + w_3$; $\quad w_3 = 0w_1 + 0w_2 + 1w_3$,

so $M = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{pmatrix}$

b) Set $V = \begin{pmatrix} -1 & 1 & 1 \\ -2 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$, $W = \begin{pmatrix} -1 & 2 & 1 \\ -3 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$. Then $T = V^{-1}W$. To get T,
Start with $(V \mid W)$ and get its RREF.

$$\left(\begin{array}{ccc|ccc} -1 & 1 & 1 & -1 & 2 & 1 \\ -2 & 3 & 1 & -3 & 3 & 1 \\ 1 & 2 & 2 & 1 & 1 & 3 \end{array}\right) \xrightarrow[r_1 + r_3]{-2r_1 + r_2} \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 3 & 3 & 0 & 3 & 4 \end{array}\right) \xrightarrow{-3r_2 + r_3} \left(\begin{array}{ccc|ccc} -1 & 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 6 & 3 & 6 & 7 \end{array}\right)$$

$$\xrightarrow[r_3/6]{-r_2 + r_1} \left(\begin{array}{ccc|ccc} -1 & 0 & 2 & 0 & 3 & 2 \\ 0 & 1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{7}{6} \end{array}\right) \xrightarrow[r_3 + r_2]{-2r_3 + r_1} \left(\begin{array}{ccc|ccc} -1 & 0 & 0 & -1 & 1 & -\frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{7}{6} \end{array}\right) \xrightarrow{-r_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{2} & 1 & \frac{7}{6} \end{array}\right)$$

Hence $T = \begin{pmatrix} 1 & -1 & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{2} & 1 & \frac{7}{6} \end{pmatrix}$

c) $P = TMT^{-1}$

d) $L(z) = (2+1+3)w_1 + (-1-3+2)w_2 + (3-2-1)w_3$
$= 6w_1 - 2w_2 + 0w_3$ in B

To get the coordinate of $L(z)$ in D, apply to $[L(z)]_B$

$[L(z)]_D = T[L(z)]_B = \begin{pmatrix} 1 & -1 & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{1}{6} \\ \frac{1}{2} & 1 & \frac{1}{6} \end{pmatrix}\begin{pmatrix} 6 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \\ 1 \end{pmatrix}$

5. [15] a) Let A and B be 8×5 matrices. If rank of A is 5, what is the dimension of $N(A)$? If the dimension of $N(B)$ is 4, what is the rank of B? (Explain each answer to get full credit.)

$\dim N(A) = 5-5=0$, by Rank-nullity Theorem

$r_B = 5-4=1$, by Rank-nullity Theorem

b) Let $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ be linearly independent vectors in $\mathbb{R}^3$. Let A be a 5×3 matrix with rank 3, and set $\mathbf{v}_1 = A\mathbf{u}_1$, $\mathbf{v}_2 = A\mathbf{u}_2$, and $\mathbf{v}_3 = A\mathbf{u}_3$. Are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ linearly independent? (Explain your answer or get no credit.)

$r_A = 3 \rightarrow \dim N(A) = 0$, by Rank-nullity Theorem; so $N(A) = \{0_{\mathbb{R}^3}\}$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$: $\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0_{\mathbb{R}^5}$. Do we have $\alpha_1 = 0 = \alpha_2 = \alpha_3$?

$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0_{\mathbb{R}^5} \Leftrightarrow \alpha_1 A u_1 + \alpha_2 A u_2 + \alpha_3 A u_3 = 0_{\mathbb{R}^5}$

$\Leftrightarrow A(\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3) = 0_{\mathbb{R}^5}$

$\Leftrightarrow \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 \in N(A) = \{0_{\mathbb{R}^3}\}$; so

$\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0_{\mathbb{R}^3}$; so $\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$, as u_1, u_2, u_3 linearly independent.

c) Complete the sentence: The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, ..., $\mathbf{u}_n$ form a basis for $\mathbb{R}^n$ when the following two conditions are met:

1) $u_1, u_2, \dots, u_n$ are linearly independent, and

2) $\mathbb{R}^n = \text{Span}(u_1, u_2, \dots, u_n)$.

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6. [10] Let $\mathbf{u}_1 = (-1, 1, 1)^T$, $\mathbf{u}_2 = (1, 2, 3)^T$ and $\mathbf{u}_3 = (2, a, b)^T$ be vectors in $\mathbb{R}^3$. For which values of a and b do we have $\mathbb{R}^3 = \text{Span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$? By Theorem 3.4.3, it is enough that u_1, u_2, u_3 be linearly independent. Now

$$\begin{vmatrix} -1 & 1 & 2 \\ 1 & 2 & a \\ 1 & 3 & b \end{vmatrix} = -(2b-3a) - (b-a) + 2(3-2)$$

$$= -3b + 4a + 2$$

So $\mathbb{R}^3 = \text{Span}(u_1, u_2, u_3)$ for a, b with $4a - 3b + 2 \neq 0$.

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7. [10] Let A and B be $n \times n$ matrices. Let r_A and r_B denote the rank of A and the rank of B respectively. Let n_A and n_B stand for the nullity of A and that of B respectively. Show that $r_A + r_B - n = n - n_A - n_B$.

$r_A + r_B - n = r_A + r_B - (r_B + n_B)$, by Rank-nullity Theorem

$= r_A + \cancel{r_B} - \cancel{r_B} - n_B$

$= n - n_A - n_B$, by Rank-nullity Theorem once more.