MAC 2313 (Calculus III)
Test 3, Thursday November 30, 2006

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.

1. [26] Evaluate each integral.

a) $\int_0^1 \int_0^\pi \int_2^3 z^2(2r+1)\sin\theta\, dz d\theta dr =$

$= \int_0^1 (2r+1)dr . \int_0^\pi \sin\theta d\theta . \int_2^3 z^2 dz$

$= r^2+r\Big]_0^1 . \Big[-\cos\theta\Big]_0^\pi . \Big[\frac{z^3}{3}\Big]_2^3$

$= 2 . 2 . \left(\frac{27-8}{3}\right)$

$= \frac{76}{3}$

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b) $\int_0^1 \int_1^2 (x^2 - y^2)\, dx dy =$

$= \int_0^1 \int_1^2 x^2\, dx dy - \int_0^1 \int_1^2 y^2 dx dy$

$= \int_0^1 \frac{x^3}{3}\Big|_1^2 dy - \int_0^1 y^2 x\Big]_1^2 dy$

$= \int_0^1 \frac{8-1}{3} dy - \int_0^1 y^2(2-1) dy$

$= \frac{7}{3}y\Big]_0^1 - \frac{y^3}{3}\Big|_0^1$

$= \frac{7}{3} - \frac{1}{3} = \frac{6}{3} = 2$

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c) $\int_0^1 \int_e^{e^2} \ln y\, dy dx = \int_0^1 \left\{ \Big[y \ln y\Big]_e^{e^2} - \int_e^{e^2} y\left(\frac{1}{y}\right) dy \right\} dx$

$= \int_0^1 \left\{ e^2 2 - e.1 - \Big[y\Big]_e^{e^2} \right\} dx$

$= 2e^2 - e - (e^2 - e)$

$= e^2$

8

d) $\int_0^2 \int_0^z \int_y^{\sqrt{y}} 2x\, dx dy dz =$

$= \int_0^2 \int_0^z x^2\Big]_y^{\sqrt{y}} dy dz$

$= \int_0^2 \int_0^z (y - y^2) dy dz$

$= \int_0^2 \frac{y^2}{2} - \frac{y^3}{3}\Big]_0^z dz$

$= \int_0^2 \frac{z^2}{2} - \frac{z^3}{3} dz$

$= \frac{z^3}{6} - \frac{z^4}{12}\Big]_0^2$

$= \frac{8}{6} - \frac{16}{12}$

$= 0$

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2. [10] Use polar coordinates to evaluate $\int_0^1 \int_0^x \sqrt{x^2+y^2}\, dy\, dx$. ($\int \sec^n x\, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x\, dx$, $\int \sec x\, dx = \ln|\sec x + \tan x| + C$.)

$x = r\cos\theta\,,\quad y = r\sin\theta$

$0 \le x \le 1 \to 0 \le r\cos\theta \le 1 \to 0 \le r \le \sec\theta$

$0 \le y \le x \to 0 \le r\sin\theta \le r\cos\theta \to \sin\theta \le \cos\theta$, $0 \le \sin\theta,\ 0 \le \cos\theta \to 0 \le \theta \le \frac{\pi}{4}$

$$= \int_0^{\pi/4}\int_0^{\sec\theta} r \cdot r\, dr\, d\theta = \int_0^{\pi/4} \frac{r^3}{3}\Big]_0^{\sec\theta} d\theta = \frac{1}{3}\int_0^{\pi/4} \sec^3\theta\, d\theta$$

$$= \frac{1}{3}\left[\left[\frac{\sec\theta\tan\theta}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4}\sec\theta\, d\theta\right] = \frac{1}{6}\left[\sqrt{2} + \ln(\sec\theta+\tan\theta)\Big]_0^{\pi/4}\right]$$

$$= \underline{\frac{\sqrt{2} + \ln(\sqrt{2}+1)}{6}}$$

3. [12] Let $F(x,y) = xy\vec{i} + yz\vec{j} + zx\vec{k}$. Find div F, and curl F.

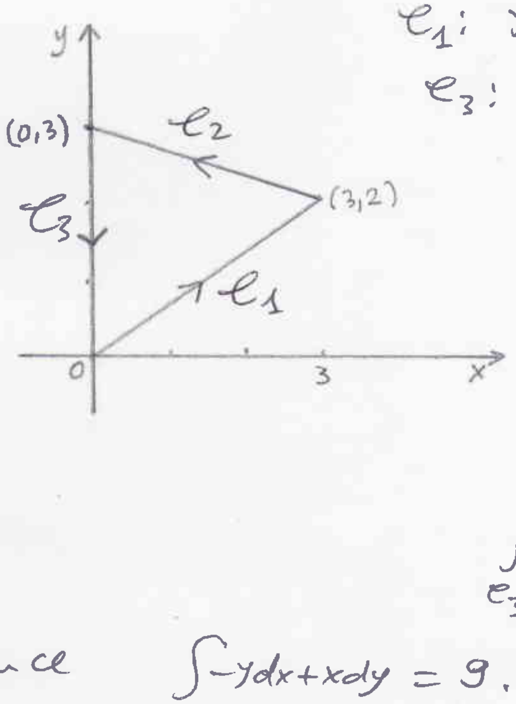
$$\operatorname{div} F = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(zx) = y + z + x \qquad 6$$

$$= x + y + z$$

$$\operatorname{Curl} F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial x & \partial y & \partial z \\ xy & yz & xz \end{vmatrix} = (0-y)\vec{i} - (z-0)\vec{j} + (0-x)\vec{k}$$

$$\operatorname{Curl} F = -y\vec{i} - z\vec{j} - x\vec{k} \qquad 6$$

4. [12] Evaluate the line integral $\int_C -y\, dx + x\, dy$ along the curve shown on the figure.



$e_1:\ y = \frac{2}{3}x$ 2 $\qquad e_2:\ y = -\frac{1}{3}x + 3$ 2

$e_3:\ x = 0$ 1

$$\int_e -y\,dx + x\,dy = \int_{e_1} -y\,dx + x\,dy + \int_{e_2} -y\,dx + x\,dy + \int_{e_3} -y\,dx + x\,dy \quad 1.$$

$$\int_{e_1} -y\,dx + x\,dy = \int_0^3 -\frac{2}{3}x\,dx + x\left(\frac{2}{3}dx\right) = 0 \quad 2$$

$$\int_{e_2} -y\,dx + x\,dy = \int_3^0 -\left(-\frac{1}{3}x+3\right)dx + x\left(-\frac{1}{3}dx\right)$$

$$= \int_3^0 -3\,dx = 3\int_0^3 dx = 3x\Big]_0^3 = 9 \quad 2$$

$$\int_{e_3} -y\,dx + x\,dy = \int_{e_3} -y(0) + 0 \cdot dy = 0 \quad 2$$

Hence $\int_e -y\,dx + x\,dy = 9.$

5. [15] Let $F(x,y) = (y\sin x - \cos y)\vec{i} + (x\sin y - \cos x)\vec{j}$. Show that F is conservative , and find a potential function φ for F. Evaluate the line integral $\int_C (y\sin x - \cos y)\,dx + (x\sin y - \cos x)\,dy$ along the curve C parametrized by $r(t) = \tan(\pi t/4)\vec{i} + \tan^{-1} t\vec{j}$, $\quad 0 \le t \le 1$.

$\frac{\partial}{\partial y}(y\sin x - \cos y) = \sin x + \sin y = \frac{\partial}{\partial x}(x\sin y - \cos x)$; so F is conservative

Hence $F = \nabla\varphi$. $\varphi_x = y\sin x - \cos y$, $\varphi_y = x\sin y - \cos x$

$\varphi = \int \varphi_x\,dx = \int (y\sin x - \cos y)\,dx = -y\cos x - x\cos y + C(y)$

$\varphi_y = -\cos x + x\sin y + C'(y) = -\cos x + x\sin y$; So $C'(y) = 0$; we may choose $C(y) = 0$, $\forall y$. so $\varphi(x,y) = -y\cos x - x\cos y$

$\varphi(x,y) = -y\cos x - x\cos y$. Now $r(0) = (0,0)$, $r(1) = (1, \frac{\pi}{4})$; hence

$\int_{\mathcal{C}} (y\sin x - \cos y)\,dx + (x\sin y - \cos x)\,dy = \varphi(1, \frac{\pi}{4}) - \varphi(0,0)$, by the FTLI.

$= -\frac{\pi}{4}\cos(1) - \cos\frac{\pi}{4} - 0$

$= -\frac{\pi}{4}\cos(1) - \frac{\sqrt{2}}{2}$.

6. [7] Use Green's theorem to evaluate the line integral $\int_C (2\tan^{-1}(y/x))\,dx + \ln(x^2+y^2)\,dy$ along the curve C parametrized by $r(t) = (3 + 2\cos t)\vec{i} + (1 + 3\sin t)\vec{j}$, $\quad 0 \le t \le 2\pi$.

$\mathcal{C}$ is the ellipse centered at $(3, 1)$; $\mathcal{C}$ is a simple closed curve

By Green's theorem

$$\int_{\mathcal{C}} 2\tan^{-1}(y/x)\,dx + \ln(x^2+y^2)\,dy$$

$$= \iint_R \left\{\frac{\partial}{\partial x}\ln(x^2+y^2) - \frac{\partial}{\partial y}\left(2\tan^{-1}(y/x)\right)\right\} dA, \quad R = \text{region enclosed by } \mathcal{C}$$

$$= \iint_R \left\{\frac{2x}{x^2+y^2} - \frac{2}{x}\cdot\frac{1}{1+\frac{y^2}{x^2}}\right\} dA$$

$$= \iint_R \left\{\frac{2x}{x^2+y^2} - \frac{2x}{x^2+y^2}\right\} dA$$

$$= 0$$

7. [18] Let $f(x,y) = xy$. Use the Lagrange multipliers to find the maximum and minimum values of f subject to the constraint: $2x^2 + 3y^2 = 1$.

Set $h(x,y) = xy - \lambda(2x^2+3y^2-1)$

$$\nabla h(x,y) = \begin{pmatrix} y - 4\lambda x \\ x - 6\lambda y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{array}{l} y = 4\lambda x \\ x = 6\lambda y. \end{array}$$

Hence

$$x = 6\lambda(4\lambda x) = 24\lambda^2 x, \text{ and } x \neq 0,$$

Since $x=0 \rightarrow y=0$, contradicts $2x^2+3y^2=1$.

So $1 = 24\lambda^2$ or $\lambda = \pm \frac{\sqrt{24}}{24} = \pm \frac{2\sqrt{6}}{24} = \pm \frac{\sqrt{6}}{12}$

- $\lambda = \frac{\sqrt{6}}{12} \rightarrow y = \frac{\sqrt{6}}{3}x$; so $2x^2 + 3\left(\frac{\sqrt{6}}{3}x\right)^2 = 1$, or
  $2x^2 + 3\left(\frac{6x^2}{9}\right) = 1$, or $4x^2 = 1 \rightarrow x = \pm\frac{1}{2}$.
  Therefore, we have the pts $\underbrace{\left(\frac{1}{2}, \frac{\sqrt{6}}{6}\right)}_{A}$, and $\underbrace{\left(-\frac{1}{2}, -\frac{\sqrt{6}}{6}\right)}_{B}$
- $\lambda = -\frac{\sqrt{6}}{12} \rightarrow y = -\frac{\sqrt{6}}{3}x$; leading to $4x^2 = 1$; so $x = \pm\frac{1}{2}$.
  Therefore we have the point $\underbrace{\left(\frac{1}{2}, -\frac{\sqrt{6}}{6}\right)}_{C}$, and $\underbrace{\left(-\frac{1}{2}, \frac{\sqrt{6}}{6}\right)}_{D}$

Now

$f\left(\frac{1}{2}, \frac{\sqrt{6}}{6}\right) = f\left(-\frac{1}{2}, -\frac{\sqrt{6}}{6}\right) = \frac{\sqrt{6}}{12}$; f has max. value at A & B

$f\left(\frac{1}{2}, -\frac{\sqrt{6}}{6}\right) = f\left(-\frac{1}{2}, \frac{\sqrt{6}}{6}\right) = -\frac{\sqrt{6}}{12}$; f has min. value at C & D.

For problem 8, see text or notes _