MAC 2312 (Calculus II) — Answers
Test 3, Wednesday October 28, 2015

Name: **PID:**

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; *guessing the correct answers won't earn you any credits.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.**

1. [12] Consider the sequence (a_n) given by $a_n = \frac{2n-1}{5n+3}$, $n = 1, 2, \dots$. a) Find a_1, a_2, a_3 and a_4. b) Use the difference $a_{n+1} - a_n$ to show that the sequence (a_n) is strictly increasing. c) Show that $a_n < \frac{2}{5}$ for all n. d) Show that the sequence (a_n) converges. e) Find its limit.

a) $a_1 = \frac{1}{8}$, $a_2 = \frac{3}{13}$, $a_3 = \frac{5}{18}$, $a_4 = \frac{7}{23}$

b) $a_{n+1} - a_n = \frac{2n+1}{5n+8} - \frac{(2n-1)}{5n+3} = \frac{(2n+1)(5n+3) - (2n-1)(5n+8)}{(5n+8)(5n+3)}$

$= (\cancel{10n^2} + \cancel{6n} + 5n + 3 - (\cancel{10n^2} + \cancel{16n} - 5n - 8))/((5n+8)(5n+3))$

$= 11/((5n+8)(5n+3)) > 0$; so (a_n) is strictly increasing

c) $a_n - \frac{2}{5} = \frac{2n-1}{5n+3} - \frac{2}{5} = \frac{5(2n-1) - 2(5n+3)}{5(5n+3)} = \frac{10n - 5 - 10n - 6}{5(5n+3)} = \frac{-11}{5(5n+3)} < 0$

So $a_n < \frac{2}{5}$ for all n.

d) (a_n) is strictly increasing, and bounded from above; so (a_n) converges

e) $\lim_{n\to+\infty} a_n = \lim_{n\to\infty} \frac{2n-1}{5n+3} = \lim_{n\to\infty} \frac{2n}{5n} = \frac{2}{5}$.

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2. [5] Decide whether each statement is true or false. No explanation needed.

a) If a sequence $(a_n)_n$ is bounded from below and from above, then it converges. False, pick $a_n = (-1)^n$, $n = 1, 2, \dots$

b) If $\lim_{k\to\infty} u_k = 0$, then the series $\sum u_k$ converges. False, pick $u_k = \frac{1}{k}$, $k = 1, 2, \dots$

c) If $\lim_{k\to\infty} \sqrt[k]{|u_k|} = \frac{100}{99}$, then the series $\sum u_k$ converges absolutely. False, by root test, as $\frac{100}{99} > 1$

d) If the series $\sum |u_k|$ converges, then the series $\sum u_k$ converges. True; absolute cvgence implies cvgence

e) If $0 < a_k \le b_k$ for all $k \ge 280$, and $\sum a_k$ diverges, then $\sum b_k$ diverges too. True by Comparison test

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3. [10] Determine whether the following series converge or diverge. Explain your answers. a) $\sum_{k=1}^{\infty} \frac{k^2+3}{2k^2-5k+7}$, b) $\sum_{k=1}^{\infty} \frac{(-1)^k 3^{2k}}{8^k}$, c) State the ratio test on the back of the page.

a) $\lim_{k\to+\infty} \frac{k^2+3}{2k^2-5k+7} = \lim_{k\to+\infty} \frac{\cancel{k^2}}{2\cancel{k^2}} = \frac{1}{2} \neq 0$; so series diverges by Divergence Test

b) Series is a geometric series with ratio $r = -\frac{9}{8}$. Now $|r| = \frac{9}{8} > 1$; so series diverges.

c) See text or notes.

4. [10] Express the nth partial sum of the infinite series $\sum_{k=0}^{\infty} \frac{1}{4k^2-9}$ as a telescoping sum, and thereby show that the infinite series converges.

$$S_n = \sum_{k=0}^{n} \frac{1}{4k^2-9} = \sum_{k=0}^{n} \frac{1}{(2k-3)(2k+3)} = \frac{1}{6}\sum_{k=0}^{n}\left(\frac{1}{2k-3} - \frac{1}{2k+3}\right) = \frac{1}{6}\sum_{k=0}^{n}\frac{1}{2k-3} - \frac{1}{6}\sum_{k=0}^{n}\frac{1}{2k+3}$$

$$= \frac{1}{6}\sum_{k=0}^{n}\frac{1}{2k-3} - \frac{1}{6}\sum_{k=3}^{n+3}\frac{1}{2k-3} = \frac{1}{6}\left(-\frac{1}{3} - 1 + 1\right) + \cancel{\frac{1}{6}\sum_{k=3}^{n}\frac{1}{2k-3}} - \cancel{\frac{1}{6}\sum_{k=3}^{n}\frac{1}{2k-3}} - \frac{1}{6}\left(\frac{1}{2n-1} + \frac{1}{2n+1} + \frac{1}{2n+3}\right)$$

$$= -\frac{1}{18} - \frac{1}{6}\left(\frac{1}{2n-1} + \frac{1}{2n+1} + \frac{1}{2n+3}\right).\ \lim_{n\to+\infty} S_n = -\frac{1}{18}.$$ The sequence of nth partial sums (S_n) converges to $-\frac{1}{18}$. So series converges with sum $S = -\frac{1}{18}$.

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5. [10] Use the ratio test to show that the series $\sum_{k=1}^{\infty} \frac{k!(-3)^k}{(2k+1)!}$ converges absolutely.

$$\lim_{k\to+\infty}\frac{|u_{k+1}|}{|u_k|} = \lim_{k\to+\infty}\frac{(k+1)!\,3^{k+1}}{(2k+3)!}\cdot\frac{(2k+1)!}{k!\,3^k} = 3\lim_{k\to+\infty}\frac{\cancel{k!}(k+1)\,(\cancel{(2k+1)!})}{\cancel{k!}\,\cancel{(2k+1)!}\,(2k+2)(2k+3)}$$

$$= 3\lim_{k\to+\infty}\frac{k+1}{(2k+2)(2k+3)} = \frac{3}{2}\lim_{k\to+\infty}\frac{1}{2k+3} = \frac{3}{2}(0) = 0;$$

So series converges absolutely by the ratio test.

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6. [10] Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{\frac{8}{9}}}$ converge? If so, does it converge absolutely or conditionally? If not, why?

Series is an alternating series with $a_k = \frac{1}{k^{8/9}}$. Now $a_k = \frac{1}{k^{8/9}} > \frac{1}{(k+1)^{8/9}}$ for all k, and $\lim_{k\to+\infty} a_k = 0$; hence series converges by the alternating series test.

On the other hand $\sum_{k=1}^{\infty}\frac{1}{k^{8/9}}$ is a p-series with $p = 8/9 < 1$; so it diverges. The initial series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k^{8/9}}$ converges conditionally.