MAC 2313 (Calculus III)
Test 3, Wednesday October 28, 2015

Name: **PID:**

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. *You will not get any credit to any of the problems if you do not show your work.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Total=65 points. Good luck.**

1. [10] a) Find the absolute minimum and absolute maximum values of the function $f(x, y, z) = x + 2y - 3z$ on the ellipsoid $x^2 + 4y^2 + 9z^2 = 6$.

Set $h(x,y,z) = x+2y-3z - \lambda(x^2+4y^2+9z^2-6)$.

$\nabla h(x,y,z) = \vec{0} \to \langle 1-2\lambda x, 2-8\lambda y, -3-18\lambda z\rangle = \langle 0,0,0\rangle$

So $x = 1/2\lambda$, $y = 1/4\lambda = \frac{x}{2}$, $z = -\frac{1}{6\lambda} = -\frac{x}{3}$

$x^2+4y^2+9z^2 = 6 \to x^2+4(\frac{x}{2})^2+9(-\frac{x}{3})^2 = 6 \to x^2+x^2+x^2=6$

So $x^2 = 2$; hence $x = \pm\sqrt{2}$. We have the points $(-\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3})$ and $(\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{3})$. Now $f(-\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}) = -\sqrt{2}-2-\sqrt{2} = -3\sqrt{2}$ is the absolute minimum of f, while $f(\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{3}) = \sqrt{2}+\sqrt{2}+\sqrt{2} = 3\sqrt{2}$ is the absolute maximum of f, on the ellipsoid.

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2. [20] a) Use spherical coordinates to find the volume of the solid G bounded above by the plane $z = 3$ and below by the cone $z = \sqrt{3x^2 + 3y^2}$. You may just setup the triple integral for the volume including all integration limits without evaluating it.

$\sqrt{3}\sqrt{x^2+y^2} \le z \le 3 \to \sqrt{3}\rho\sin\phi \le \rho\cos\phi \le 3$

$\tan\phi \le \frac{1}{\sqrt{3}}$ $\Downarrow$ $0 \le \phi \le \frac{\pi}{6}$; $\rho \le 3\sec\phi$

$$V = \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{3\sec\phi} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

b) Use cylindrical coordinates to find the volume of the solid G bounded above by the sphere $x^2+y^2+z^2 = 12$ and below by the paraboloid $z = x^2 + y^2$.

intersection: $r^2+r^4 = 12 \to r^4+r^2-12 = 0 \to \underbrace{(r^2-3)}_{=0}\underbrace{(r^2+4)}_{>0}=0$

$\left.\begin{array}{l} r^2 = 3 \\ r \ge 0\end{array}\right) \to r = \sqrt{3}$

$$V = \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{r^2}^{\sqrt{12-r^2}} r\,dz\,dr\,d\theta = 2\pi\int_0^{\sqrt{3}}(r\sqrt{12-r^2} - r^3)dr$$

$$= 2\pi\left[-\frac{(12-r^2)^{3/2}}{3} - \frac{r^4}{4}\right]_0^{\sqrt{3}} = 2\pi\left[\frac{12\sqrt{12}}{3} - \frac{3(\sqrt{3})^4}{3} - \frac{8+1}{4}\right]$$

$$= 2\pi\left(8\sqrt{3} - \frac{45}{4}\right)$$

3. [8] Write down a triple integral equivalent to the integral $\int_0^2 \int_0^{4-z^2} \int_0^z \frac{\cos(2y)}{4-y}\, dxdydz$, by integrating first in z, then in x and finally in y. Do not attempt to evaluate any of the integrals.

$0 \le z \le 2 \qquad 0 \le y \le 4 - z^2 \to z^2 \le 4-y \to z \le \sqrt{4-y}$

$0 \le x \le z \le \sqrt{4-y}$

$$\int_0^2 \int_0^{4-z^2} \int_0^z \frac{\cos(2y)}{4-y}\, dxdydz = \int_0^4 \int_0^{\sqrt{4-y}} \int_x^{\sqrt{4-y}} \frac{\cos(2y)}{4-y}\, dzdx$$

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4. [12] Evaluate each integral

a) $\int_1^2 \int_0^{\ln x} e^{x+y}\, dydx = \int_1^2 \left[e^{x+y}\right]_0^{\ln x} dx$

$= \int_1^2 (xe^x - e^x)dx$

$= xe^x\Big]_1^2 - \int_1^2 e^x dx - \int_1^2 e^x dx$

$= (x-2)e^x\Big]_1^2$

$= 0 - (-1)e$

$= e$

b) $\int_0^1 \int_y^{y^2} \int_0^{\ln x} ye^z\, dzdxdy = \int_0^1 y \int_y^{y^2} e^z\Big]_0^{\ln x} dxdy$

$= \int_0^1 y \int_y^{y^2} (x-1)dxdy$

$= \int_0^1 y\left(\frac{x^2}{2} - x\right)\Big]_y^{y^2} dy$

$= \int_0^1 y\left(\frac{y^4}{2} - \frac{y^2}{2} - y^2 + y\right)dy$

$= \frac{y^6}{12} - \frac{3}{8}y^4 + \frac{y^3}{3}\Big]_0^1$

$= \frac{1}{12} - \frac{3}{8} + \frac{1}{3}$

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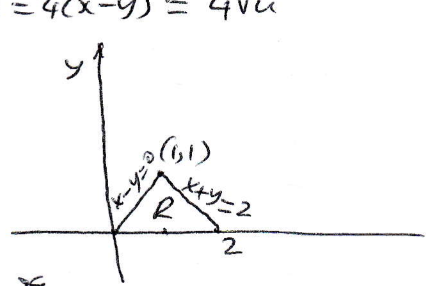
5. [15] a) If $u = (x-y)^2$ and $v = x+y$, find the Jacobian $\partial(x,y)/\partial(u,v)$ and express it in terms of u and v, assuming $x > y$.

b) Use an appropriate change of variables to evaluate $\int\int_R \frac{(x-y)^2}{x+y}\, dydx$, where R is the triangular region with vertices at (0,0), (1,1) and (2,0).

a) $\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2(x-y) & -2(x-y) \\ 1 & 1 \end{vmatrix} = 2(x-y) + 2(x-y) = 4(x-y) = 4\sqrt{u}$

Hence $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4\sqrt{u}}$



b) $\iint_R \frac{(x-y)^2}{x+y}\, dydx$

$= \int_0^2 \int_0^{v^2} \frac{u}{v} \frac{1}{4\sqrt{u}}\, dudv$

$= \frac{1}{4}\int_0^2 \frac{1}{v} \int_0^{v^2} \sqrt{u}\, dudv$

$= \frac{1}{4}\int_0^2 \frac{1}{v} \frac{2}{3} u^{3/2}\Big]_0^{v^2} dv$

$= \frac{1}{6}\int_0^2 \frac{1}{v}(v^3)dv = \frac{1}{6}\int_0^2 v^2 dv = \frac{1}{6} \frac{v^3}{3}\Big]_0^2$

$= \frac{1}{6}\left(\frac{8}{3}\right) = \frac{4}{9}.$

If (x,y) inside of R, then

$x - y \ge 0$ or $u \ge 0$

$0 \le x+y \le 2$

$0 \le v \le 2$

Now $x - y \le x + y$,

so $(x-y)^2 \le (x+y)^2$.

or $u \le v^2$