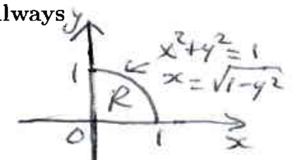
MAC 2313 (Calculus III) — Answers
Test 3, Wednesday November 23, 2016

Name: **PID:**

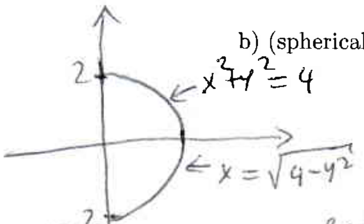
Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question. *You will not get any credit to any of the problems if you do not show your work.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Always do your best. Total=85 points.**

1. [13] Evaluate each integral: a) (polar coordinates) $\int_0^1 \int_0^{\sqrt{1-y^2}} \sin(\pi(x^2+y^2))\,dx\,dy =$



$$\int_0^{\frac{\pi}{2}}\int_0^1 r\sin(\pi r^2)\,dr\,d\theta$$
$$= \frac{\pi}{2}\left[-\frac{\cos(\pi r^2)}{2\pi}\right]_0^1 = \frac{\pi}{2}\left(-\frac{\cos(\pi)+1}{2\pi}\right) = \frac{2}{4} = \frac{1}{2}$$

b) (spherical coordinates) $\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} \int_0^{\sqrt{4-x^2-y^2}} \frac{1}{x^2+y^2+z^2+1}\,dz\,dx\,dy =$ $\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2}\int_0^2 \frac{\rho^2\sin\phi}{\rho^2+1}\,d\rho\,d\phi\,d\theta$



projection of solid on xy-plane: $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ or $0\le\theta\le\frac{\pi}{2}$ or $\frac{3\pi}{2}\le\theta\le 2\pi$

$0\le z\le\sqrt{4-x^2-y^2}$

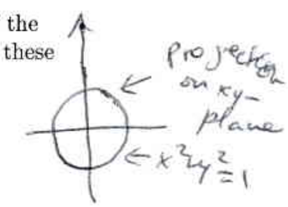
$0\le\phi\le\frac{\pi}{2}$ $\quad \rho\cos\phi\le\sqrt{4-\rho^2\sin^2\phi}$

$\rho^2\cos^2\phi\le 4-\rho^2\sin^2\phi$

$\rho^2(\cos^2\phi+\sin^2\phi)\le 4 \to \rho^2\le 4 \Longleftrightarrow \rho\le 2$

$$= \pi\int_0^{\pi/2}\sin\phi\cdot\int_0^2 \frac{\rho^2+1-1}{\rho^2+1}\,d\rho$$
$$= \pi\left[-\cos\phi\right]_0^{\pi/2}\int_0^2 1-\frac{1}{\rho^2+1}\,d\rho$$
$$= \pi\left[\rho-\tan^{-1}\rho\right]_0^2$$
$$= \pi(2-\tan^{-1}(2))$$

2. [12] a) Use cylindrical coordinates to evaluate the volume of the solid bounded above by the cone $z = 2-\sqrt{x^2+y^2}$ and below by the paraboloid $z = x^2+y^2$. b) Write down the coordinates $(\bar{x}, \bar{y}, \bar{z})$ of the centroid of that solid as iterated triple integrals including all limits of integration, but do not evaluate these integrals.



At Intersection: $2-r=r^2$, $r=\sqrt{x^2+y^2}$

$r^2+r-2=0$

$(r+2)(r-1)=0$

$r=1$

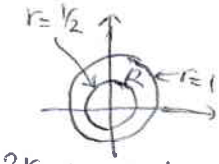
$$V=\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r} r\,dz\,dr\,d\theta$$
$$= 2\pi\int_0^1 r(2-r-r^2)\,dr = 2\pi\left[r^2-\frac{r^3}{3}-\frac{r^4}{4}\right]_0^1$$
$$= 2\pi\left(1-\frac{1}{3}-\frac{1}{4}\right) = 2\pi\left(1-\frac{7}{12}\right) = \frac{10\pi}{12} = \frac{5\pi}{6}$$

$$\bar{x} = \frac{6}{5\pi}\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r} r\cdot r\cos\theta\,dz\,dr\,d\theta$$

$$\bar{y} = \frac{6}{5\pi}\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r} r\cdot r\sin\theta\,dz\,dr\,d\theta$$

$$\bar{z} = \frac{6}{5\pi}\int_0^{2\pi}\int_0^1\int_{r^2}^{2-r} rz\,dz\,dr\,d\theta$$

3. [6] Evaluate the surface integral $\iint_\sigma z^2 dS$ if σ is the portion of the cone $z = 2\sqrt{x^2+y^2}$ between $z = 1$ and $z = 2$.



$z_x = \frac{2x}{z}$, $z_y = \frac{2y}{z}$, $z_x^2 + z_y^2 + 1 = 1 + \frac{4(x^2+y^2)}{z^2} = 1 + \frac{4(x^2+y^2)}{4(x^2+y^2)} = 1+1 = 2$

$2r = 1 \to r = 1/2$
$2r = 2 \to r = 1$

$\iint_\sigma z^2 ds = \int_0^{2\pi}\int_{1/2}^{1} \sqrt{2}\, 4r^2 \cdot r\, dr\, d\theta = 2\sqrt{2}\pi\, [r^4]_{1/2}^{1}$
$= 2\pi\sqrt{2}\left(1 - \frac{1}{16}\right) = \frac{15\pi\sqrt{2}}{8}$

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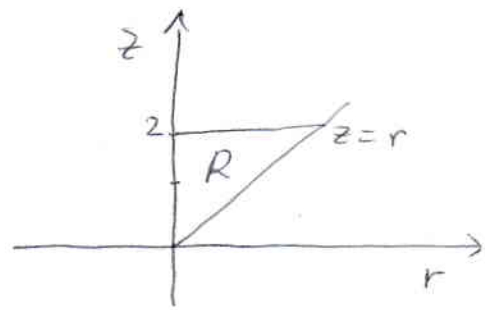
4. [8] Find the mass of the lamina represented by the parametric surface $\vec{r}(u,v) = u\cos v\,\vec{i} + u\,\vec{j} + u\sin v\,\vec{k}$ if $1 \le u \le 2$ and $0 \le v \le \pi$, and its density is $\delta(x,y,z) = \sqrt{x^2+y^2+z^2}$.

$M = \iint_\sigma \delta(x,y,z)\, dS$, $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & 1 & \sin v \\ -u\sin v & 0 & u\cos v \end{vmatrix} = u\cos v\,\vec{i} - u(\cos^2 v + \sin^2 v)\vec{j} + (0 + u\sin v)\vec{k}$
$= u\cos v\,\vec{i} - u\vec{j} + u\sin v\,\vec{k}$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{u^2\cos^2 v + u^2 + u^2\sin^2 v}$
$= \sqrt{u^2(\cos^2 v + \sin^2 v) + u^2} = \sqrt{2u^2} = \sqrt{2}\,u$; $x^2+y^2+z^2 = 2u^2 = \delta^2(x,y,z)$

$M = \int_0^\pi \int_1^2 \sqrt{2u^2}\,\sqrt{2}\,u\, du\, dv = 2\pi\left[\frac{u^3}{3}\right]_1^2 = 2\pi\left[\frac{7}{3}\right] = \frac{14\pi}{3}$

5. [10] Reverse the order of integration and evaluate the integral:



$\int_0^2 \int_r^2 r\sqrt{r^2+z^2}\, dz dr = \int_0^2 \int_0^z r\sqrt{r^2+z^2}\, dr\, dz$

$= \int_0^2 \frac{1}{3}(r^2+z^2)^{3/2}\Big]_0^z dz$

$= \frac{1}{3}\int_0^2 (2z^2)^{3/2} - z^3\, dz$

$= \frac{(2\sqrt{2}-1)}{3}\int_0^2 z^3 dz$

$= \frac{2\sqrt{2}-1}{3}\,\frac{z^4}{4}\Big]_0^2 = \frac{4}{3}(2\sqrt{2}-1)$

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6. [10] a) Find $\operatorname{div}\mathbf{F}(x,y,z)$ and $\operatorname{curl}\mathbf{F}(x,y,z)$ if $\mathbf{F}(x,y,z) = y^2x\,\vec{i} - z^2y\,\vec{j} + x^2z\,\vec{k}$.

b) Find the Jacobian $\dfrac{\partial(x,y)}{\partial(u,v)}$ of the transformation $u = 2x - y^2$, $v = xy$.

a) $\operatorname{div} F(x,y,z) = \partial_x(y^2x) + \partial_y(-z^2y) + \partial_z(x^2z) = y^2 - z^2 + x^2$

$\operatorname{curl} F(x,y,z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ y^2x & -z^2y & x^2z \end{vmatrix} = (0 - (-2zy))\vec{i} - (2xz - 0)\vec{j} + (0 - 2yx)\vec{k}$
$= 2yz\,\vec{i} - 2xz\,\vec{j} - 2xy\,\vec{k}$

$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 2 & -2y \\ y & x \end{vmatrix} = 2x + 2y^2$; so $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{2x+2y^2}$

7. [15] Let $F(x,y) = (2x\sin y + y^3 e^x)\vec{i} + (x^2\cos y + 3y^2 e^x)\vec{j}$. a) Show that F is conservative. b) Find a potential function φ for F. c) Evaluate the line integral $\int_C (2x\sin y + y^3e^x)\,dx + (x^2\cos y + 3y^2e^x)\,dy$ along the curve C parametrized by $\vec{r}(t) = \ln(1+t)\vec{i} + \tan^{-1}t\,\vec{j}$, $\quad 0 \le t \le 1$.

a) $\partial_y(2x\sin y + y^3e^x) = 2x\cos y + 3y^2e^x = \partial_x(x^2\cos y + 3y^2e^x)$; so F is conservative

b) (i) $\varphi_x = 2x\sin y + y^3e^x$, (ii) $\varphi_y = x^2\cos y + 3y^2e^x$

Integrate (i) w.r.t. x:

(iii) $\varphi(x,y) = \int 2x\sin y + y^3e^x\,dx = x^2\sin y + y^3e^x + k(y)$

Differentiate (iii) w.r.t. y:

$\varphi_y(x,y) = x^2\cos y + 3y^2e^x + k'(y)$

$= x^2\cos y + 3y^2e^x$, by (ii)

So $k'(y) = 0$; $k(y) = C = \text{constant}$; choose $C = 0$

$\varphi(x,y) = x^2\sin y + y^3e^x$

c) $\int_C \vec{F}\cdot d\vec{r} = \varphi(\ln 2, \frac{\pi}{4}) - \varphi(\ln 1, \tan^{-1}0)$

$= \frac{\sqrt{2}}{2}(\ln 2)^2 + (\frac{\pi}{4})^3(2) - \varphi(0,0) = \frac{\sqrt{2}}{2}(\ln 2)^2 + \frac{\pi^3}{32}$

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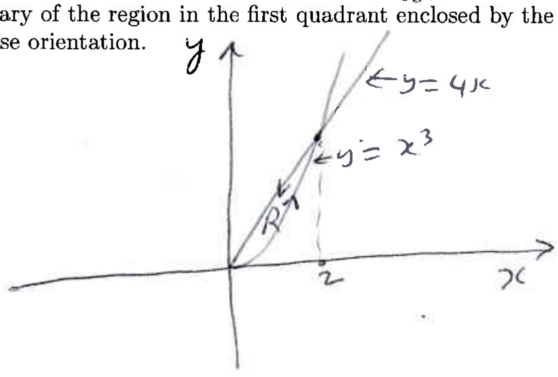
8. [11] a) State Green's Theorem. See text or notes

b) Use Green's Theorem to evaluate the line integral $K = \oint_C (2xy + e^{\cos x} - 3y)\,dx + (x^2 + \ln(1+\sin^2 y))\,dy$, where C is the boundary of the region in the first quadrant enclosed by the curves $y = 4x$ and $y = x^3$ with a counterclockwise orientation.



$4x = x^3 \rightarrow x(4 - x^2) = 0$

$\rightarrow x = 0$ or $x = 2$ or $\cancel{x = -2}$

$K = \int_0^2\int_{x^3}^{4x} 2x - 2x + 3\,dy\,dx$

$= 3\int_0^2 4x - x^3\,dx$

$= 3\left[2x^2 - \frac{x^4}{4}\right]_0^2$

$= 3(8-4) = 12.$