MAC 2313 (Calculus III) — Answers
Test 3, Friday June 02, 2017

Name: PID:

Remember that no documents or calculators are allowed during the exam. Be as precise as possible in your work. Guessing the correct answers won't give you any credits. You must show all your work to deserve the full mark assigned to any question. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. You may use the back of the page to show your work. 2 pages. Total=60 points. Always do your best.

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1. [10] a) Find the point Q on the circle $x^2 + y^2 = 4$ that is closest to the point $P(6, 2)$. b) Find the distance between Q and P.

Let (x,y) be the coordinates of an arbitrary point on the circle

$d((x,y),P) = \sqrt{(6-x)^2+(2-y)^2} = \sqrt{(x-6)^2+(y-2)^2}$. Set $f(x,y) = (x-6)^2+(y-2)^2$

We shall find the minimum value of f on the circle.

Set $L(x,y) = (x-6)^2+(y-2)^2 - \lambda(x^2+y^2-4)$. $\nabla L(x,y) = \langle 2(x-6)-2\lambda x, 2(y-2)-2\lambda y \rangle$

$\nabla L(x,y) = \vec{0} \to x(1-\lambda) = 6 \to x = \frac{6}{1-\lambda}$, and $y(1-\lambda) = 2 \to y = \frac{2}{1-\lambda}$; note λ cannot be 1 nor $x=0$ or $y=0$

So $\frac{36}{(1-\lambda)^2} + \frac{4}{(1-\lambda)^2} = 4$; $\frac{40}{(1-\lambda)^2} = 4 \to (1-\lambda)^2 = 10 \to 1-\lambda = \pm\sqrt{10}$

Note that P being in the first quadrant, Q must lie in the first quadrant too; so $Q = \left(\frac{6}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$

• $1-\lambda = \sqrt{10} \to x = \frac{6}{\sqrt{10}}$, $y = \frac{2}{\sqrt{10}}$.

b) $d(P,Q) = \sqrt{(6-6/\sqrt{10})^2+(2-\frac{2}{\sqrt{10}})^2} = \sqrt{40(1-\frac{1}{\sqrt{10}})^2} = 2\sqrt{10}(1-\frac{1}{\sqrt{10}}) = 2(\sqrt{10}-1)$

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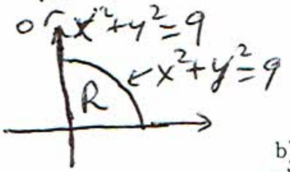
2. [20] a) Use spherical coordinates to find the volume of the solid G bounded above by the plane $z = 3$ and below by the portion of the cone $z = \sqrt{x^2 + y^2}$ that lies in the first octant.

$z = \rho\cos\phi$,

At $z = 3$,

$\sqrt{x^2+y^2} = 3$

or $x^2+y^2 = 9$



$\sqrt{x^2+y^2} = \rho\sin\phi \le \rho\cos\phi \le 3$

$\sin\phi \le \cos\phi$

$\tan\phi \le 1$

$0 \le \phi \le \frac{\pi}{4}$

Projection of solid on xy-plane is the region in Q1 enclosed by the circle $x^2+y^2 = 9$; so $0 \le \theta \le \frac{\pi}{2}$

$$V = \int_0^{\frac{\pi}{2}}\int_0^{\pi/4}\int_0^{3\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

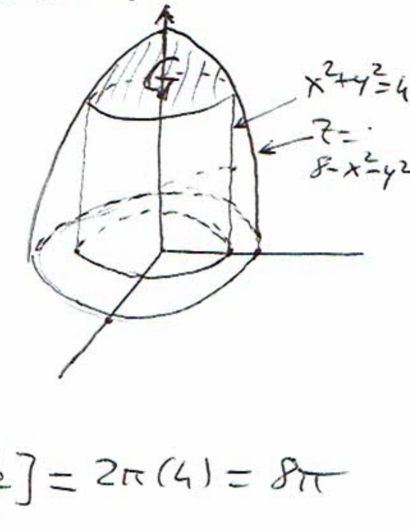
$$= \frac{\pi}{2}\int_0^{\pi/4} \frac{\rho^3}{3}\Big]_0^{3\sec\phi} \sin\phi \, d\phi = \frac{2\pi}{3}\int_0^{\pi/4} 27\tan\phi\sec^2\phi \, d\phi$$

$$= \frac{18\pi}{4}\frac{\tan^2\phi}{2}\Big]_0^{\pi/4} = \frac{9\pi}{4}(\underbrace{\tan^2\tfrac{\pi}{4}}_{1} - \underbrace{\tan^2 0}_{0}) = \frac{9\pi}{4}$$

b) Use cylindrical coordinates to find the volume of the solid G that lies inside the paraboloid $z = 8 - x^2 - y^2$ and above the cylinder $x^2 + y^2 = 4$.

At intersection, $z = 8 - 4 = 4$, Since $x^2+y^2 = 4$

The projection on the xy-plane is the region enclosed by the circle $x^2+y^2 = 4$ or $r = 2$;



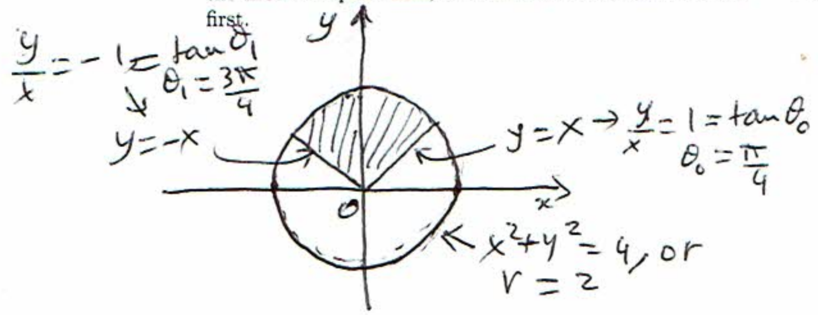
So
$$V = \int_0^{2\pi}\int_0^2\int_4^{8-r^2} r \, dz \, dr \, d\theta$$

$$= 2\pi\int_0^2 r(8-r^2-4)\,dr$$

$$= 2\pi\int_0^2 4r - r^3\,dr$$

$$= 2\pi\left[2r^2 - \frac{r^4}{4}\right]_0^2 = 2\pi\left[2(4) - \frac{16}{4}\right] = 2\pi(4) = 8\pi$$

3. [8] Express the integral $\iint_{\mathcal{R}} \sqrt{x^2+y^2}\,dA$ in polar coordinates, but do not evaluate it. $\mathcal{R}$ is the region that lies in the first two quadrants, and is enclosed by the curves $y=-x$, $y=x$, and $x^2+y^2=4$. You must sketch the region $\mathcal{R}$ first.



$$\iint_{R} \sqrt{x^2+y^2}\,dA = \int_{\pi/4}^{3\pi/4}\int_0^2 r\cdot r\,dr\,d\theta$$

4. Evaluate each integral [12]

a) $\int_2^4 \int_0^x \frac{x}{x^2+y^2}\,dydx = \int_2^4 \int_0^1 \frac{\cancel{x^2}\,du}{\cancel{x^2}(1+u^2)}\,dx$

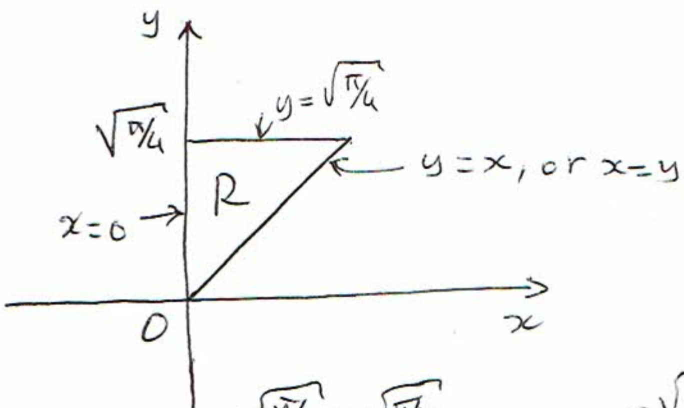
$y = ux$
$dy = x\,du$
$y=0 \to u=0$
$y=x \to u=1$

$$= 2\int_0^1 \frac{du}{1+u^2}$$
$$= 2\tan^{-1}u\Big]_0^1$$
$$= 2(\underbrace{\tan^{-1}1}_{\frac{\pi}{4}} - \underbrace{\tan^{-1}0}_{0})$$
$$= 2(\pi/4) = \frac{\pi}{2}$$

b) $\int_1^3 \int_y^{y^2} \int_0^{\ln x} ye^z\,dzdxdy = \int_1^3\int_y^{y^2} y\,e^z\Big]_0^{\ln x} dxdy$

$$= \int_1^3\int_y^{y^2} y\,(e^{\ln x} - e^0)\,dxdy$$
$$= \int_1^3\int_y^{y^2} y\,(x-1)\,dxdy$$
$$= \int_1^3 y\,\frac{(x-1)^2}{2}\Big]_y^{y^2} dy$$
$$= \int_1^3 y\,\frac{(y^2-1)^2}{2} - y\,\frac{(y-1)^2}{2}\,dy$$
$$= \frac{(y^2-1)^3}{12} - \left(\frac{y^4}{8} - \frac{y^3}{3} + \frac{y^2}{4}\right)\Big]_1^3$$
$$= \frac{8^3}{12} - \left(\frac{80}{8} - \left(\frac{26}{3}\right) + \frac{8}{4}\right)$$
$$= \frac{128+26}{3} - 12 = \frac{118}{3}$$

5. [10, **Bonus**] Use an appropriate order of integration to evaluate the double integral $\int_0^{\sqrt{\frac{\pi}{4}}}\int_x^{\sqrt{\frac{\pi}{4}}} \sec^2(y^2)\,dydx$. You must sketch the region first.



$$\int_0^{\sqrt{\pi/4}}\int_x^{\sqrt{\pi/4}} \sec^2(y)\,dydx = \int_0^{\sqrt{\pi/4}}\int_0^{y} \sec^2(y)\,dxdy$$
$$= \int_0^{\sqrt{\pi/4}} y\sec^2(y^2)\,dy$$
$$= \frac{\tan(y^2)}{2}\Big]_0^{\sqrt{\pi/4}} = \frac{\tan(\pi/4) - \tan(0)}{2}$$
$$= \frac{1}{2}$$