MAC 2312 (Calculus II) — *Answers*
Test 4, Monday November 30, 2015

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; *guessing the correct answers won't earn you any credits.* **Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Good luck.**

1. [8] Find the first four nonzero terms of the Maclaurin series for the function $f(x) = e^x \sin(x^2)$.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

$$\sin(x^2) = x^2 - \frac{x^6}{6} + \cdots$$

$$e^x \sin(x^2) = x^2 + x^3 + \frac{x^4}{2} + \frac{x^5}{6} + \cdots$$

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2. [10] a) Use a well-known Maclaurin series to find the Maclaurin series for $\cos(2x^3)$.

$$\cos(2x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k (2x^3)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k 4^k x^{6k}}{(2k)!}$$

b) Use a Maclaurin series to evaluate the integral:

$$\int_0^{\pi^2} \sin(\sqrt{x})\, dx = \int_0^{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^k (\sqrt{x})^{2k+1}}{(2k+1)!} dx$$

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$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \int_0^{\pi^2} x^{\frac{2k+1}{2}} dx$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \left[\frac{2}{2k+3} (\sqrt{x})^{2k+3}\right]_0^{\pi^2} = \sum_{k=0}^{\infty} \frac{2(-1)^k \pi^{2k+3}}{(2k+1)!(2k+3)}$$

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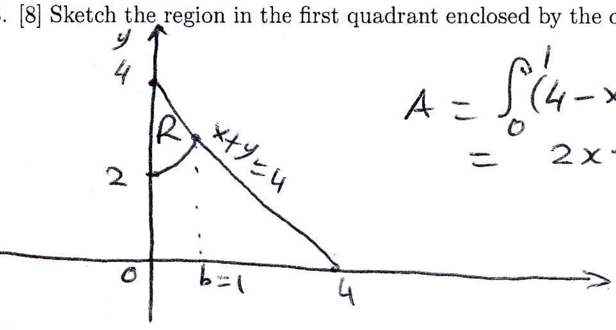
3. [8] Sketch the region in the first quadrant enclosed by the curves $y = x^2 + 2$, $x + y = 4$, $x = 0$, and find its area.

For b, solve
$x^2 + 2 = 4 - x$
$x^2 + x - 2 = 0$
$(x-1)(x+2) = 0$
$x = 1$, as $x \geq 0$



$$A = \int_0^1 (4 - x - x^2 - 2)\, dx$$

$$= \left. 2x - \frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = 2 - \frac{1}{2} - \frac{1}{3} = 2 - \frac{5}{6} = \frac{7}{6}$$

4. [10] Find the radius of convergence and the interval of convergence of the power series $\sum_{k=0}^{\infty} \frac{(-1)^k 4^k (x-7)^{2k}}{3k+4}$.

$$\rho(x) = \lim_{k\to+\infty} \frac{|(-1)^{k+1}4^{k+1}(x-7)^{2k+2}|}{3k+7} \cdot \frac{3k+4}{|(-1)^k 4^k (x-7)^{2k}|}$$

$$= \lim_{k\to+\infty} 4(x-7)^2 \frac{(3k+4)}{3k+7}$$

$$= 4(x-7)^2 < 1 \to (x-7)^2 < \frac{1}{4} \to |x-7| < \frac{1}{2}; \text{ So } R = \frac{1}{2}$$

@ $x = 7+\frac{1}{2}$: $\sum_{k=0}^{\infty} \frac{(-1)^k 4^k (\frac{1}{2})^{2k}}{3k+4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+4}$, since $4^k(\frac{1}{2})^{2k} = 4^k(\frac{1}{4^k}) = 1$

@ $x = 7-\frac{1}{2}$; $\sum_{k=0}^{\infty} \frac{(-1)^k 4^k (-\frac{1}{2})^{2k}}{3k+4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{3k+4}$, since $4^k(-\frac{1}{2})^{2k} = 4^k(\frac{1}{4^k}) = 1$

So series converges at both endpoints by A.S.T. Hence

$$I_c = \left[\frac{13}{2}, \frac{15}{2}\right]$$

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5. [8] Find the fourth Taylor polynomial about $x = \pi/3$ for $f(x) = \cos(3x)$

$f'(x) = -3\sin(3x)$, $f''(x) = -9\cos(3x)$, $f^{(3)}(x) = 27\sin(3x)$, $f^{(4)}(x) = 81\cos(3x)$

$$P_4(x) = f(\pi/3) + f'(\pi/3)(x-\pi/3) + \frac{f''(\pi/3)}{2}(x-\frac{\pi}{3})^2 + \frac{f^{(3)}(\pi/3)}{6}(x-\frac{\pi}{3})^3 + \frac{f^{(4)}(\pi/3)}{24}(x-\frac{\pi}{3})^4$$

$$= -1 + 0(x-\frac{\pi}{3}) + \frac{9}{2}(x-\frac{\pi}{3})^2 + 0\cdot(x-\frac{\pi}{3})^3 + \frac{27}{8}(x-\frac{\pi}{3})^4$$

$$= -1 + \frac{9}{2}(x-\frac{\pi}{3})^2 - \frac{27}{8}(x-\frac{\pi}{3})^4$$

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6. [12] a) Find the volume of the solid generated when the region enclosed by the curves $y = \frac{1}{1+x^2}$, $x = 0$ and $x = 1$ is revolved about the x-axis.

$$V = \pi \int_0^1 \frac{1}{(1+x^2)^2}dx, \quad \tan u = x \to dx = \sec^2 u\, du$$

$$= \pi \int_0^{\pi/4} \frac{1}{(1+\tan^2 u)^2}\sec^2 u\, du = \pi \int_0^{\pi/4} \frac{\sec^2 u}{\sec^4 u} du = \pi \int_0^{\pi/4} \sec^{-2} u\, du$$

$$= \pi \int_0^{\pi/4} \cos^2 u\, du = \pi \int_0^{\pi/4} \frac{1+\cos(2u)}{2} du = \pi\left[\frac{u}{2} + \frac{\sin(2u)}{4}\right]_0^{\pi/4}$$

$$= \frac{\pi}{4}\left(\frac{\pi}{2}+1\right)$$

b) Use the remainder estimation theorem to find an interval containing $x = 0$ over which $f(x) = \sin x$ can be approximated by $p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ to four decimal place accuracy throughout the interval.

$P(x) = P_6(x) = x - \frac{x^3}{6!} + \frac{x^5}{5!}$,

$|R_6(x)| \leq \frac{|x|^7}{7!}$

$f'(x) = \cos x$, $f''(x) = -\sin x$, $f^{(3)}(x) = -\cos x$
$f^{(4)}(x) = \sin x$, $f^{(5)}(x) = \cos x$, $f^{(6)}(x) = -\sin x$
$f^{(7)}(x) = -\cos x$

$\max_{-\infty < x < +\infty} |f^{(7)}(x)| = 1$

for $|f(x) - P_6(x)| \leq 5\times 10^{-5}$, it suffices that

$\frac{|x|^7}{7!} \leq 5\times 10^{-5}$ or $|x| \leq \sqrt[7]{5(10^{-5})7!}$

$$I = \left[-\sqrt[7]{5(10^{-5})(7!)}, \sqrt[7]{5(10^{-5})(7!)}\right]$$