MAC 2313 (Calculus III) — Key
Test 4, Monday April 20, 2015

Name: PID:

Remember that no documents or calculators are allowed during the test. Be as precise as possible in your work; you shall show all your work to deserve the full mark assigned to any question; answers without any explanation won't get any credits. Do not cheat, otherwise I will be forced to give you a zero and report your act of cheating to the University Administration. Total =65 points.Good luck.

1. [12] Use cylindrical coordinates to find the volume and centroid of the solid bounded below by the cone $z = \sqrt{x^2+y^2}$, and above by the paraboloid $z = 6 - x^2 - y^2$.

At the intersection, we have $r = 6 - r^2$, by setting $r^2 = x^2+y^2$
so $r^2 + r - 6 = 0$ or $(r+3)(r-2) = 0$; since $r \geq 0$, $r+3 > 0$; so $r = 2$ or $x^2+y^2=4$

$$\text{so } V = \int_0^{2\pi}\int_0^2\int_r^{6-r^2} r\,dz\,dr\,d\theta = 2\pi\int_0^2 r(6-r^2) - r^2\,dr = 2\pi\left[-\frac{(6-r^2)^2}{4} - \frac{r^3}{3}\right]_0^2$$

$$= 2\pi\left(\frac{6^2-2^2}{4} - \frac{2^3}{3}\right) = 2\pi\left(8 - \frac{8}{3}\right) = \frac{32\pi}{3}.$$ If C.G $= (\bar{x}, \bar{y}, \bar{z})$,

then $\bar{x} = \bar{y} = 0$, by symmetry (z-axis is an axis of symmetry for the solid)

$$\bar{z} = \frac{3}{32\pi}\int_0^{2\pi}\int_0^2\int_r^{6-r^2} rz\,dz\,dr\,d\theta = \frac{6\pi}{32\pi}\int_0^2 r\left[\frac{z^2}{2}\right]_r^{6-r^2} = \frac{3}{16}\int_0^2 \frac{r((6-r^2)^2) - r^3}{2}\,dr = \frac{3}{32}\left[-\frac{(6-r^2)^3}{6} - \frac{r^4}{4}\right]_0^2$$

$$= \frac{3}{32}\left(\frac{6^3-2^3}{6} - \frac{2^4}{4}\right) = \frac{23}{8}$$

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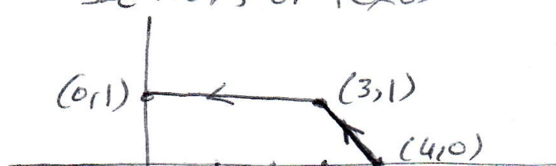
2. [8] a) State Green's Theorem. See Notes or text.



b) Can we use it to evaluate the line integral $\int_C (y^3 + \ln(1 + e^{(x^5)}))\,dx + x3^{(y^4)}\,dy$, where C is composed of the segment of line joining (4,0) to (3,1) and the segment of line joining (3,1) to (0,1)? Clearly explain your answer, but do not evaluate the line integral. The functions to integrate are continuously differentiable everywhere in the plane. But the curve C is not closed although C is simple and piecewise smooth with counterclockwise orientation. Green's Theorem cannot be applied.

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3. [10] a) Use the divergence theorem to find the flux Φ of the vector $F(x,y) = x\vec{i} + y\vec{j} + z\vec{k}$ across σ if σ is the surface of the solid G bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2+y^2}$ with outward orientation.

$\text{div}\,\vec{F}(x,y,z) = \partial_x(x) + \partial_y(y) + \partial_z(z) = 1+1+1 = 3$. At intersection $r = 4$ or $x^2+y^2 = 16$

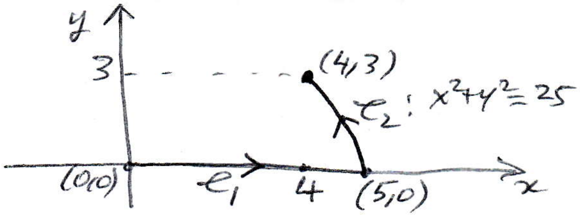
$$\Phi = \iint_\sigma \vec{F}\cdot\vec{n}\,dS = \iiint_G \text{div}\,F\,dV = \iiint_G 3\,dV$$
$$= 3\int_0^{2\pi}\int_0^4\int_r^4 r\,dz\,dr\,d\theta$$
$$= 6\pi\int_0^4 r(4-r)\,dr = 6\pi\left[2r^2 - \frac{r^3}{3}\right]_0^4$$
$$= 6\pi\left(32 - \frac{64}{3}\right) = \frac{6\pi(32)}{3} = 64\pi$$

b) Describe the portion of the sphere $x^2+y^2+z^2 = 4$ in the <u>first octant</u> in terms of the parameters θ and ϕ where (ρ, θ, ϕ) are spherical coordinates

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

Now $\rho^2 = 4$; so $\rho = 2$, $x \geq 0$, $y \geq 0$, $z \geq 0$ so $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \frac{\pi}{2}$

Hence
$x = 2\sin\phi\cos\theta$
$y = 2\sin\phi\sin\theta$
$z = 2\cos\phi$



4. [10] Evaluate the line integral: a) $\int_C (x^2+xy)\,dx - 2xy\,dy$, where C is composed of the line segment from (0,0) to (5,0) and the arc of the circle $x^2+y^2=25$ from (5,0) to (4,3).

On C_1: $y=0$, so $\int_{C_1}(x^2+xy)dx - 2xy\,dy = \int_{C_1} x^2 dx = \int_0^5 x^2 dx = \frac{x^3}{3}\Big]_0^5 = \frac{125}{3}$
$0 \le x \le 5$

On C_2: $y=\sqrt{25-x^2}$, $dy = \frac{-x}{\sqrt{25-x^2}}dx$; $\int_{C_2}(x^2+xy)dx - 2xy\,dy = \int (x^2+x\sqrt{25-x^2})dx + \frac{2x^2\sqrt{25-x^2}}{\sqrt{25-x^2}}dx$

x goes from 5 to 4

$= -\int_4^5 (3x^2 + x\sqrt{25-x^2})dx$

$= -x^3 + (25-x^2)^{3/2}/3 \Big]_5^4$

$= 4^3 - 5^3 + 0 - \frac{9^{3/2}}{3}$

Hence $\int_C (x^2+xy)dx - 2xy\,dy = \frac{125}{3} + 64 - 125 - 9$

b) Evaluate the line integral $\int_C \frac{xye^{(x^2)}}{x^2+y^2}\,ds$, where C is the parametric curve given by $\vec r(t) = 2\cos t\,\vec i + 2\sin t\,\vec j$, $0 \le t \le \pi$.

$ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt = \sqrt{(-2\sin t)^2 + (2\cos t)^2}\,dt = \sqrt{4(\sin^2 t + \cos^2 t)}\,dt = \sqrt{4}\,dt = 2\,dt$

Similarly, $x^2+y^2 = 4$. so $\int_C \frac{xye^{(x^2)}}{x^2+y^2} = \int_0^\pi \frac{2(2\cos t\; 2\sin t)e^{4\cos^2 t}}{4}\,dt$

$= -\frac{e^{4\cos^2 t}}{4}\Big]_0^\pi = \frac{e^4 - e^4}{4} = 0$

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5. [15] Let $F(x,y) = (2x\cos y + e^{2x})\vec i - (x^2 \sin y + e^{-y})\vec j$. a) Show that F is conservative. b) Find a potential function φ for F. c) Evaluate the line integral $\int_C (2x\cos y + e^{2x})\,dx - (x^2\sin y + e^{-y})\,dy$ along the curve C parametrized by $\vec r(t) = \ln(1+t)\,\vec i + \sin^{-1} t\,\vec j$, $0 \le t \le 1$.

a) $\partial_y(2x\cos y + e^{2x}) = -2x\sin y = \partial_x[-(x^2\sin y + e^{-y})]$; so F is conservative

b) (i) $\varphi_x = 2x\cos y + e^{2x}$, (ii) $\varphi_y = -x^2\sin y - e^{-y}$

Integrate (i) w.r.t. x: (iii) $\varphi = \int (2x\cos y + e^{2x})dx = x^2\cos y + \frac{e^{2x}}{2} + k(y)$

Differentiate (iii) w.r.t. y: (iv) $\varphi_y = -x^2\sin y + k'(y) = -x^2\sin y - e^{-y}$, by (ii)

So $k'(y) = -e^{-y}$ and $k(y) = \int -e^{-y}dy = e^{-y} + c$; we may choose $c=0$.

So $\varphi(x,y) = x^2\cos y + \frac{e^{2x}}{2} + e^{-y}$.

c) $t=0 \to (\ln 1, \sin^{-1} 0) = (0,0)$, $t=1 \to (\ln 2, \sin^{-1} 1) = (\ln 2, \frac{\pi}{2})$. Hence

$\int_C (\,) = \varphi(\ln 2, \frac{\pi}{2}) - \varphi(0,0)$ by FTLI

$= (\ln 2)^2\cos\frac{\pi}{2} + \frac{e^{2\ln 2}}{2} + e^{-\frac{\pi}{2}} - (0 + \frac{1}{2} + 1)$

$= 0 + \frac{4}{2} + e^{-\frac{\pi}{2}} - \frac{3}{2}$

6. [10] a) Let $\mathsf{F}(x,y,z) = (3x^2 - 2yz)\vec i + (4y^2 - 2xz)\vec j + (2z^2 - 2xy)\vec k$. Find div$\mathsf{F}$ and curl$\mathsf{F}$.

$\operatorname{div} F = \partial_x(3x^2 - 2yz) + \partial_y(4y^2 - 2xz) + \partial_z(2z^2 - 2xy)$

$= 6x + 8y + 4z$

$\operatorname{curl} F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \partial_x & \partial_y & \partial_z \\ 3x^2-2yz & 4y^2-2xz & 2z^2-2xy \end{vmatrix} = (-2x + 2x)\vec i - (-2y + 2y)\vec j + (-2z + 2z)\vec k$

$= 2(x - x)\vec i + 2(y - y)\vec j + 2(z - z)\vec k$

$= \vec 0$

b) Set up, but do not evaluate, one iterated integral equal to the surface integral $\int\int_\sigma z^2 x\,dS$, where σ is the portion of the cylinder $x^2+y^2=6$ in the first octant between the planes $z=0$, $z=4$, $y=x$ and $x=3y$. Remember to include all integration limits.

Method 1: project σ on xz. $\partial_x(x^2+y^2) = \partial_x(6) = 0$

$2x + 2y_x y = 0 \to y_x = -\frac{x}{y}$, $y_z = 0$

$\sqrt{y_x^2 + y_z^2 + 1} = \sqrt{\frac{x^2}{y^2} + 1} = \sqrt{\frac{x^2+y^2}{y^2}} = \frac{\sqrt 6}{y} = \frac{\sqrt 6}{\sqrt{6-x^2}}$.

$\iint_\sigma z^2 x\,dS = \int_0^4 \int_{\sqrt 3}^{3\sqrt{\frac{3}{5}}} \frac{z^2 x\sqrt 6}{\sqrt{6-x^2}}\,dx\,dz$

$y \le x \le 3y$

$y^2 \le x^2 \le 9y^2$

$x^2+y^2 \le 2x^2 \to 6 \le 2x^2 \to \sqrt{3} \le x$

$x^2 + 9x^2 \le 9(y^2+x^2) = 9(6) = 54$

$10x^2 \le 54 \to x \le 3\sqrt{\frac{3}{5}}$

$2y^2 \le x^2+y^2 = 6 \to y \le \sqrt 3$

$x^2+y^2 \le 9y^2 + y^2 = 10y^2$

$6 \le 10y^2 \to \sqrt{\frac{3}{5}} \le y$

Method 2: Project σ on yz. $\sqrt{x_y^2 + x_z^2 + 1} = \frac{\sqrt 6}{\sqrt{6-y^2}}$

$\iint_\sigma z^2 x\,dS = \int_0^4 \int_{\sqrt{\frac{3}{5}}}^{\sqrt 3} z^2 \cancel{\sqrt{6-y^2}}\,\frac{\sqrt 6}{\cancel{\sqrt{6-y^2}}}\,dy\,dz$