Stabilization of a transmission system involving thermoelasticity

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Overview

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- The transmission system.

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- Well-posedness and strong stability.

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- Well-posedness and strong stability.
- Exponential stability.
- Polynomial stability

Consider the following system

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\begin{aligned}
&y_{tt} - a\Delta y + \alpha\Delta\theta = 0 \text{ in } \Omega \times (0, \infty) \\
&\theta_t - \mu\Delta\theta + \beta y_t = 0 \text{ in } \Omega \times (0, \infty) \\
&y = 0, \quad \theta = 0 \text{ on } \Gamma \times (0, \infty) \\
&y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad \theta(x,0) = \theta^0(x) \text{ in } \Omega,
\end{aligned}
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Ω = bounded domain in $\mathbb{R}^N$ with smooth boundary, a and μ are positive constants, and α, β are constants with $\alpha\beta > 0$.

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Ω = bounded domain in $\mathbb{R}^N$ with smooth boundary, a and μ are positive constants, and α, β are constants with $\alpha\beta > 0$.

System is well-posed in $H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$, and its energy given by

$$E(t) = \frac{1}{2}\int_\Omega \{|y_t(x,t)|^2 + a|\nabla y(x,t)|^2 + \frac{\alpha}{\beta}|\nabla\theta(x,t)|^2\}\, dx$$

is a nonincreasing function of the time variable t, as

$$E'(t) = -\frac{\mu\alpha}{\beta}\int_\Omega |\Delta\theta(x,t)|^2\, dx, \quad \text{a.e. } t > 0.$$

It can be shown that the semigroup associated with this system is exponentially stable, but not analytic.

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Question: Knowing that this thermoelasticity system is exponentially stable, how robust is this stability? In other words, if this structure is connected, through the usual transmission conditions, to another undamped structure modeled by a wave equation with possibly a different speed of propagation, is the exponential stability property kept?

Problem formulation

Consider the transmission system

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\begin{array}{l}
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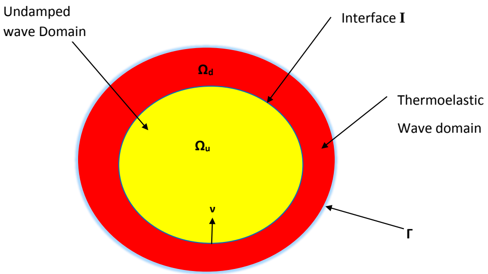
\theta_t - \mu\Delta\theta + \beta y_t = 0 \text{ in } \Omega_d \times (0,\infty) \\
y = 0, \quad \theta = 0 \text{ on } \Gamma \times (0,\infty) \\
y(x,0) = y^0(x), \quad y_t(x,0) = y^1(x), \quad \theta(x,0) = \theta^0(x) \text{ in } \Omega_d, \\
z_{tt} - b\Delta z = 0 \text{ in } \Omega_u \times (0,\infty) \\
z = y, \quad b\partial_\nu z = a\partial_\nu y \text{ on } I \times (0,\infty) \\
z(x,0) = z^0(x), \quad z_t(x,0) = z^1(x) \text{ in } \Omega_u,
\end{array}
$$

where a, b and μ are positive constants and α and β are constants with $\alpha\beta > 0$, while ν denotes the unit outward normal to the boundary of Ω_d. The initial data are given in appropriate Hilbert spaces to be specified later on.
We are interested in the study of stability issues for this system.

Geometric configuration



Some literature

This work was inspired by closely related works on

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- structural acoustics (Avalos-Lasiecka,...)

Well-posedness and strong stability

Set $V = \{(u, w) \in H^1(\Omega_d) \times H^1(\Omega_u); u = 0 \text{ on } \Gamma, \quad u = w \text{ on } I\}$, and introduce the Hilbert space over the field $\mathbb{C}$ of complex numbers $\mathcal{H} = V \times L^2(\Omega_d) \times L^2(\Omega_u) \times H_0^1(\Omega_d)$, equipped with the norm

$$||Z||_{\mathcal{H}}^2 = \int_{\Omega_d} \{a|\nabla u|^2 + |v|^2 + \frac{\alpha}{\beta}|\nabla \varphi|^2\}\, dx + \int_{\Omega_u} \{b|\nabla w|^2 + |z|^2\}\, dx,$$

$$\forall Z = (u, w, v, z, \varphi) \in \mathcal{H}.$$

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$$||Z||^2_{\mathcal{H}} = \int_{\Omega_d} \{a|\nabla u|^2 + |v|^2 + \frac{\alpha}{\beta}|\nabla\varphi|^2\}\, dx + \int_{\Omega_u} \{b|\nabla w|^2 + |z|^2\}\, dx,$$

$$\forall Z = (u, w, v, z, \varphi) \in \mathcal{H}.$$

Setting $Z = (y, y', \theta, z, z')$, the system may be recast as:

$$Z' - \mathcal{A}Z = 0 \text{ in } (0, \infty), \quad Z(0) = (y^0, y^1, \theta^0, z^0, z^1),$$

Well-posedness and strong stability

the unbounded operator $\mathcal{A} : D(\mathcal{A}) \longrightarrow \mathcal{H}$ is given by

$$\mathcal{A} = \begin{pmatrix} 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ a\Delta & 0 & 0 & 0 & -\alpha\Delta \\ 0 & b\Delta & 0 & 0 & 0 \\ 0 & 0 & -\beta I & 0 & \mu\Delta \end{pmatrix}$$

with

$$D(\mathcal{A}) = \Big\{(u, w, v, z, \varphi) \in V \times V \times H_0^1(\Omega_d); a\Delta u - \alpha\Delta\varphi \in L^2(\Omega_d) \\ \Delta w \in L^2(\Omega_u), \quad \mu\Delta\varphi - \beta v \in H_0^1(\Omega_d), \\ \text{and } a\partial_\nu u = b\partial_\nu w \text{ on } I\Big\}.$$

Well-posedness and strong stability

Theorem 1

Suppose that Ω_d and Ω_u have Lipschitz boundaries, and assume that $\text{meas}(\partial\Omega_d \cap \partial\Omega_u) \neq 0$. The operator $\mathcal{A}$ generates a C_0-semigroup of contractions $(S(t))_{t\geq 0}$ on $\mathcal{H}$, which is strongly stable:

$$\lim_{t\to\infty} ||S(t)Z^0||_{\mathcal{H}} = 0, \quad \forall Z^0 \in \mathcal{H}.$$

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On the other hand, one checks that the operator $\mathcal{A}$ has a compact resolvent; so the spectrum $\sigma(\mathcal{A})$ is discrete.
Next, one shows that $\mathcal{A}$ has no purely imaginary eigenvalue. The stability theorem in Arendt-Batty (1988) yields the claimed strong stability result.

Exponential stability

Theorem 2

Suppose that Ω_d and Ω_u have C^2 boundaries, and $b > a$. Further assume that Ω_d is a collar around Ω_u, and Ω_u is strictly star-shaped with respect to some $x^0 \in \mathbb{R}^N$:

$$\exists \rho > 0 : (x - x^0) \cdot \nu(x) \leq -\rho \text{ for all } x \text{ on I .}$$

The semigroup $(S(t))_{t\geq 0}$ is exponentially stable; more precisely, there exist positive constants M and γ with

$$||S(t)Z^0||_{\mathcal{H}} \leq M \exp(-\gamma t)||Z^0||_{\mathcal{H}}, \quad \forall Z^0 \in \mathcal{H}.$$

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Then apply a theorem due to Prüss or Huang on exponential decay of bounded semigroups.

We shall prove that there exists a constant $C > 0$ such that for every $U = (f, g, h, k, l)$ in $\mathcal{H}$, the element $Z = (i\lambda - \mathcal{A})^{-1}U = (u, w, v, z, \varphi)$ in $D(\mathcal{A})$ satisfies:

$$||Z||_{\mathcal{H}} \leq C||U||_{\mathcal{H}}, \quad \forall \lambda \in \mathbb{R}.$$

The equation

$$(i\lambda - \mathcal{A})Z = U \tag{1}$$

may be recast as

$$\begin{cases} i\lambda u - v = f \text{ in } \Omega_d \\ i\lambda z - w = g \text{ in } \Omega_u \\ i\lambda v - a\Delta u + \alpha\Delta\varphi = h \text{ in } \Omega_d \\ i\lambda z - bw = k \text{ in } \Omega_u \\ i\lambda\varphi - \mu\Delta\varphi + \beta v = l \text{ in } \Omega_d. \end{cases}$$

It easily follows from the equation (1)

$$\frac{\alpha\mu}{\beta}\int_{\Omega_d} |\Delta\varphi(x)|^2\, dx = \Re\left((i\lambda - \mathcal{A})Z, Z\right) = \Re(U, Z) \leq ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}.$$

Using appropriate multipliers and Green's formula, one derives

$$
\begin{aligned}
&\int_{\Omega_d} (|v|^2 + a|\nabla u|^2)\, dx \\
&= \frac{2}{\beta}\Re \int_{\Omega_d} \{(\mu\Delta\varphi + l)\bar{v} - a\nabla\bar{u}\cdot\nabla\varphi + \alpha|\nabla\varphi|^2 + \bar{h}\varphi\}\, dx \\
&\quad + \Re \int_{\Omega_d} \{v\bar{f} + \alpha\nabla\bar{u}\cdot\nabla\varphi + h\bar{u}\}\, dx + a\int_I (\partial_\nu u)\bar{u}\, d\Gamma, \\
&\le C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + a\int_I (\partial_\nu u)\bar{u}\, d\Gamma,
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&\leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + a\int_I (\partial_\nu u)\bar{u}\, d\Gamma,
\end{aligned}$$

Set $m(x) = x - x^0$. With Green's formula, we have the identity:

$$\Re \int_{\Omega_d} (i\lambda v - a\Delta u)(2m \cdot \nabla \bar{u} + (N-1)\bar{u})\, dx$$

$$= \int_{\Omega_d} \{|v|^2 + a|\nabla u|^2 - v(2m \cdot \nabla \bar{f} + (N-1)\bar{f})\}\, dx - a \int_{\Gamma} (m \cdot \nu)|\partial_\nu u|^2\, d\Gamma$$

$$- \int_I \{(m \cdot \nu)|v|^2 + a(\partial_\nu u)(2m \cdot \nabla \bar{u} + (N-1)\bar{u}) - a(m \cdot \nu)|\nabla u|^2\}\, d\Gamma$$

$$= \Re \int_{\Omega_d} (h - \alpha\Delta\varphi)(2m \cdot \nabla \bar{u} + (N-1)\bar{u})\, dx.$$

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$$= \Re \int_{\Omega_d} (h - \alpha\Delta\varphi)(2m \cdot \nabla \bar{u} + (N-1)\bar{u})\, dx.$$

Similarly, one has:

$$\Re\int_{\Omega_u} k(2m \cdot \nabla \bar{w} + (N-1)\bar{w})dx = \Re\int_{\Omega_u} (i\lambda z - a\Delta w)(2m \cdot \nabla \bar{w} + (N-1)\bar{w})dx$$

$$= \int_{\Omega_u} \{|z|^2 + b|\nabla w|^2 - z(2m \cdot \nabla \bar{k} + (N-1)\bar{k})\}\, dx$$

$$+ \int_I \{(m \cdot \nu)|z|^2 + b(\partial_\nu w)(2m \cdot \nabla \bar{w} + (N-1)\bar{w}) - b(m \cdot \nu)|\nabla w|^2\}\, d\Gamma.$$

Using Cauchy-Schwarz and Poincaré inequalities, it follows from those identities:

$$\int_{\Omega_d}(|v|^2 + a|\nabla u|^2 + \frac{\alpha}{\beta}|\nabla\varphi|^2)\,dx + \int_{\Omega_u}(|z|^2 + b|\nabla w|^2)\,dx$$

$$\leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + \frac{a}{b}(b-a)\int_I (m\cdot\nu)|\partial_\nu u|^2\,d\Gamma$$

$$+ (b-a)\int_I (m\cdot\nu)|\nabla_\tau u|^2\,d\Gamma + \int_\Gamma (m\cdot\nu)|\partial_\nu u|^2\,d\Gamma.$$

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$$+ (b-a)\int_I (m\cdot\nu)|\nabla_\tau u|^2\,d\Gamma + \int_\Gamma (m\cdot\nu)|\partial_\nu u|^2\,d\Gamma.$$

Thanks to the geometric constraint on Ω_u, and $b > a$, we get

$$||Z||_{\mathcal{H}}^2 + \int_I |\partial_\nu u|^2\,d\Gamma \leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}})$$
$$+ C_0\int_\Gamma |\partial_\nu u|^2\,d\Gamma.$$

Let $q \in [\mathcal{C}^1(\Omega_d)]^N$ be a vector field satisfying $q = \nu$ on Γ and $q = 0$ on I. Multiplier techniques show that:

$$\int_\Gamma |\partial_\nu u|^2 \, d\Gamma \leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}) + C_0 \int_{\Omega_d} (|v|^2 + a|\nabla u|^2) \, dx$$

$$\leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}}) + C_0 \left| \int_I (\partial_\nu u) \bar{u} \, d\Gamma \right|$$

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Using the preceding estimate, we derive

$$\int_\Gamma |\partial_\nu u|^2 \, d\Gamma \leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + C_0 \int_I |u|^2 \, d\Gamma.$$

By interpolation, one derives

$$C_0 \int_I |u|^2 \, d\Gamma \le \varepsilon \int_{\Omega_d} |\nabla u|^2 \, dx + C_\varepsilon \int_{\Omega_d} |u|^2 \, dx, \quad \forall \varepsilon > 0.$$

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Hence

$$||Z||_{\mathcal{H}}^2 + \lambda^2 \int_{\Omega_d} |u|^2 \, dx \leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}} ||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}} ||Z||_{\mathcal{H}})$$

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$$+ C_0 \varepsilon ||Z||_{\mathcal{H}}^2 + C_\varepsilon \int_{\Omega_d} |u|^2 \, dx.$$

Choosing an appropriate ε, and using Young inequality, one derives

$$||Z||_{\mathcal{H}} \leq C_0 ||U||_{\mathcal{H}}$$

provided $|\lambda| > \lambda_0$ for some suitable $\lambda_0 > 0$.

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Remark. Case: $a = b$. Following Rauch-Zhang-Zuazua, we set $\psi = y 1_{\Omega_d} + z 1_{\Omega_u}$, and recast the transmission system as

$$\begin{aligned}
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&y(x,0) = y^0(x)1_{\Omega_d} + z^0(x)1_{\Omega_u} \quad \psi_t(x,0) = y^1(x)1_{\Omega_d} + z^1(x)1_{\Omega_u}, \text{ in } \Omega,\\
&\theta(x,0) = \theta^0(x) \text{ in } \Omega_d,
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where $\Omega = \Omega_d \cup \bar{\Omega}_u$, Using the continuity of the resolvent for $|\lambda| \leq \lambda_0$, we get the claimed estimate. $\square$

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\end{aligned}$$

where $\Omega = \Omega_d \cup \bar{\Omega}_u$,
with (Ω_d, T) satisfying the Bardos-Lebeau-Rauch geometric control condition for some $T > 0$:
every ray of geometric optics enters Ω_d in a time less than T.

Following Lebeau ideas, one derives the observability estimate:

$$E(0) \leq C \int_0^T \int_{\Omega_d} \{r(t)^2 |y_t(x,t)|^2 + |\Delta\theta(x,t)|^2\} \, dxdt,$$

for some large enough T and an appropriate cut-off function r.

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Hence

$$E(t) \leq \gamma E(0), \quad \forall t \geq T$$

for $\gamma = \frac{C}{C+1} < 1$.

Following Lebeau ideas, one derives the observability estimate:

$$E(0) \leq C \int_0^T \int_{\Omega_d} \{r(t)^2 |y_t(x,t)|^2 + |\Delta\theta(x,t)|^2\} \, dxdt,$$

for some large enough T and an appropriate cut-off function r.
Using appropriate multipliers, one can then get rid of the term involving y_t, obtaining:

$$E(0) \leq C \int_0^T \int_{\Omega_d} |\Delta\theta(x,t)|^2 \, dxdt.$$

Hence

$$E(t) \leq \gamma E(0), \quad \forall t \geq T$$

for $\gamma = \frac{C}{C+1} < 1$.
The semigroup property can then be invoked to claim the exponential decay of the energy.

Polynomial stability

Theorem 3

Suppose that Ω_d and Ω_u have C^2 boundaries. Further assume that Ω_d is a collar around Ω_u, and $a > b$. There exists a positive constant C such that the semigroup $(S(t))_{t\geq 0}$ satisfies:

$$||S(t)Z^0||_{\mathcal{H}} \leq \frac{C||Z^0||_{D(\mathcal{A})}}{(1+t)^{\frac{1}{4}}}, \quad \forall Z^0 \in D(\mathcal{A}), \quad \forall t \geq 0.$$

Proof Sketch.

Following the proof of Theorem 2, we already have:

$$\int_{\Omega_d}(|v|^2 + a|\nabla u|^2 + \frac{\alpha}{\beta}|\nabla\varphi|^2)\,dx + \int_{\Omega_u}(|z|^2 + b|\nabla w|^2)\,dx$$
$$\le C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + \frac{a}{b}(b-a)\int_I (m\cdot\nu)|\partial_\nu u|^2\,d\Gamma$$
$$+ (b-a)\int_I (m\cdot\nu)|\nabla_\tau u|^2\,d\Gamma + \int_\Gamma (m\cdot\nu)|\partial_\nu u|^2\,d\Gamma.$$

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$$+ (b-a)\int_I (m\cdot\nu)|\nabla_\tau u|^2\,d\Gamma + \int_\Gamma (m\cdot\nu)|\partial_\nu u|^2\,d\Gamma.$$

Now, one checks

$$\int_I |\nabla_\tau u|^2\,d\Gamma \leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + C_0\int_I \{|\partial_\nu u|^2 + |u|^2\}\,d\Gamma$$

Hence, as earlier, and for $|\lambda|$ large enough:

$$||Z||_{\mathcal{H}}^{2} \leq C_0(||U||_{\mathcal{H}}^{\frac{1}{2}}||Z||_{\mathcal{H}}^{\frac{3}{2}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + C_0 \int_I |\partial_\nu u|^2 \, d\Gamma.$$

Hence, as earlier, and for $|\lambda|$ large enough:

$$||Z||^2_{\mathcal{H}} \leq C_0(||U||^{\frac{1}{2}}_{\mathcal{H}}||Z||^{\frac{3}{2}}_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}) + C_0 \int_I |\partial_\nu u|^2 \, d\Gamma.$$

Now borrowing ideas from Avalos-Triggiani (EECT, 2(2013)), one derives:

$$\int_I |\partial_\nu u|^2 \, d\Gamma \leq C_0(\lambda^2 + 1)(||U||^{\frac{1}{2}}_{\mathcal{H}}||Z||^{\frac{3}{2}}_{\mathcal{H}} + ||U||_{\mathcal{H}}||Z||_{\mathcal{H}}).$$

Thanks to Young inequality, we finally get:

$$||Z||^2_{\mathcal{H}} \leq C_0\lambda^8||U||^2_{\mathcal{H}}.$$

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Thanks to Young inequality, we finally get:

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The claimed polynomial decay then follows from a Theorem of Tomilov and Borichev. □

And if anyone thinks that he knows anything, he knows nothing yet as he ought to know.

THANKS!